Certainty Equivalent Control for Markov Jump Systems

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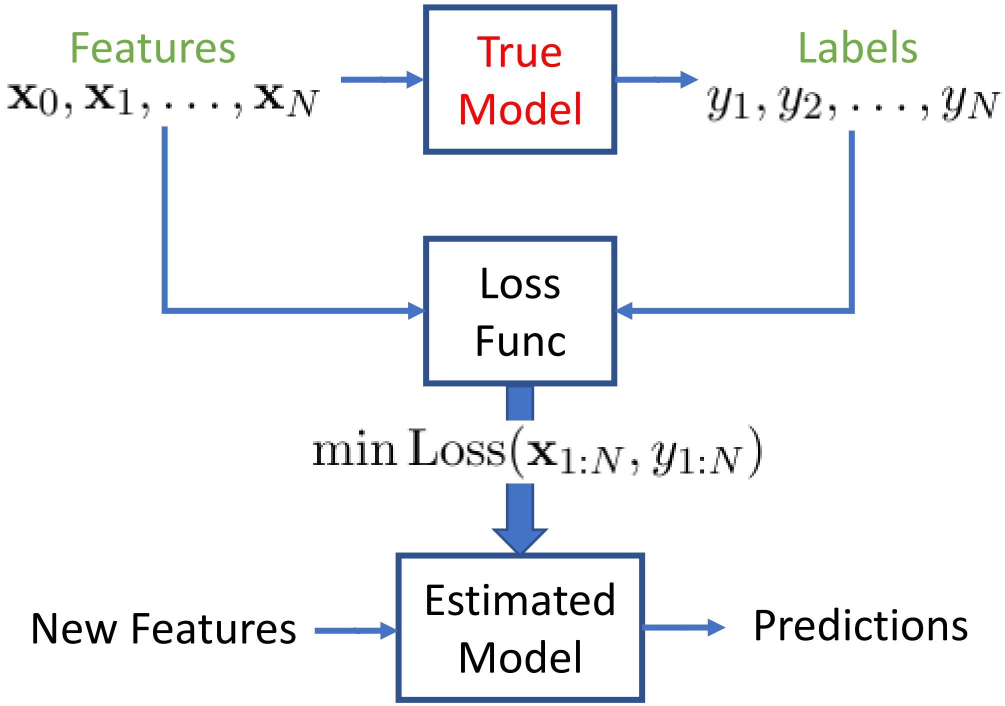
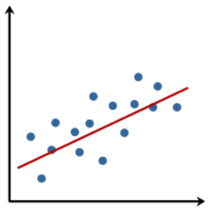
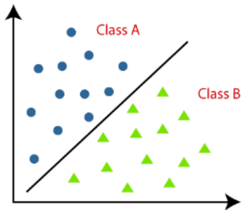
ML vs Control







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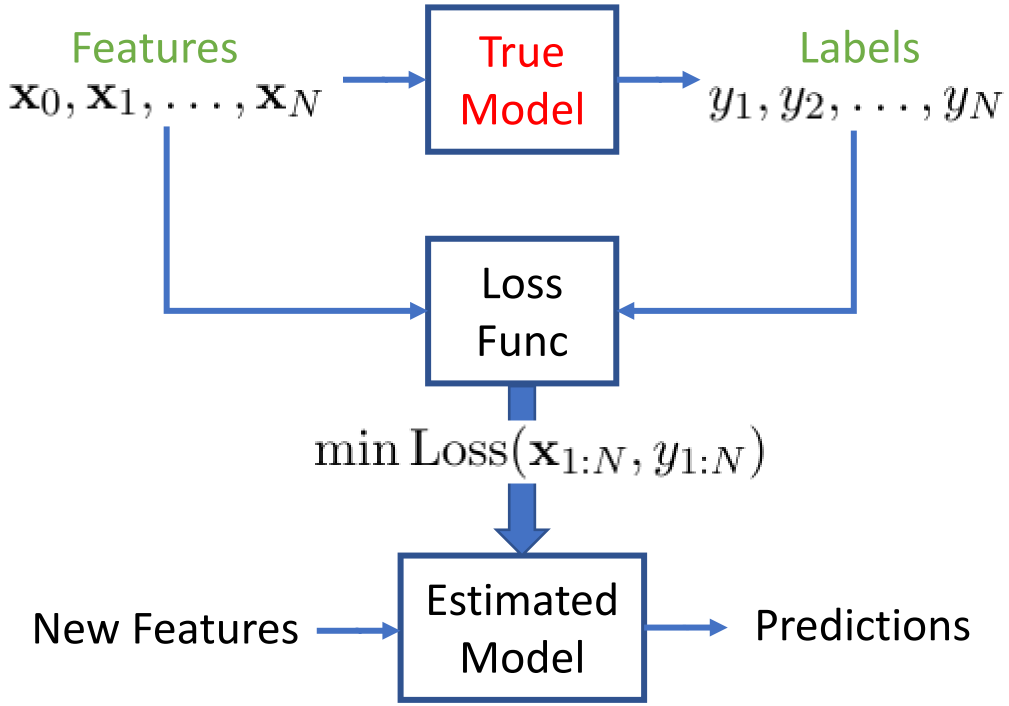
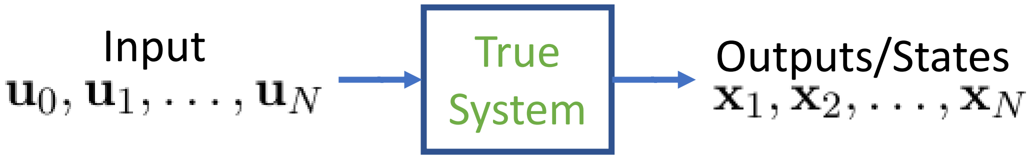
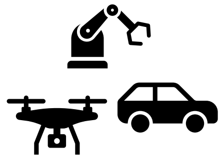
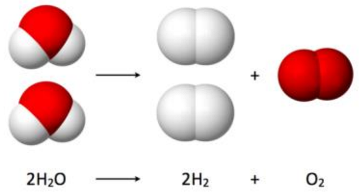
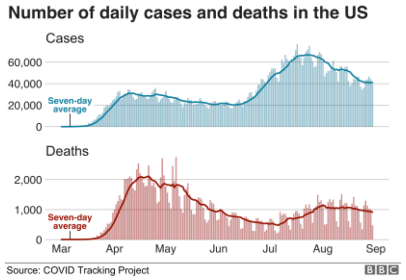
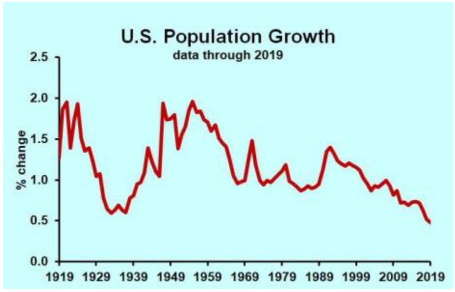


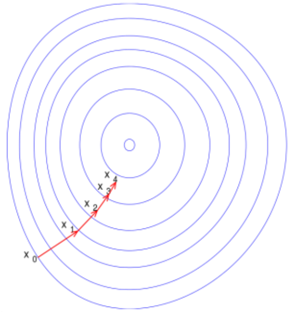




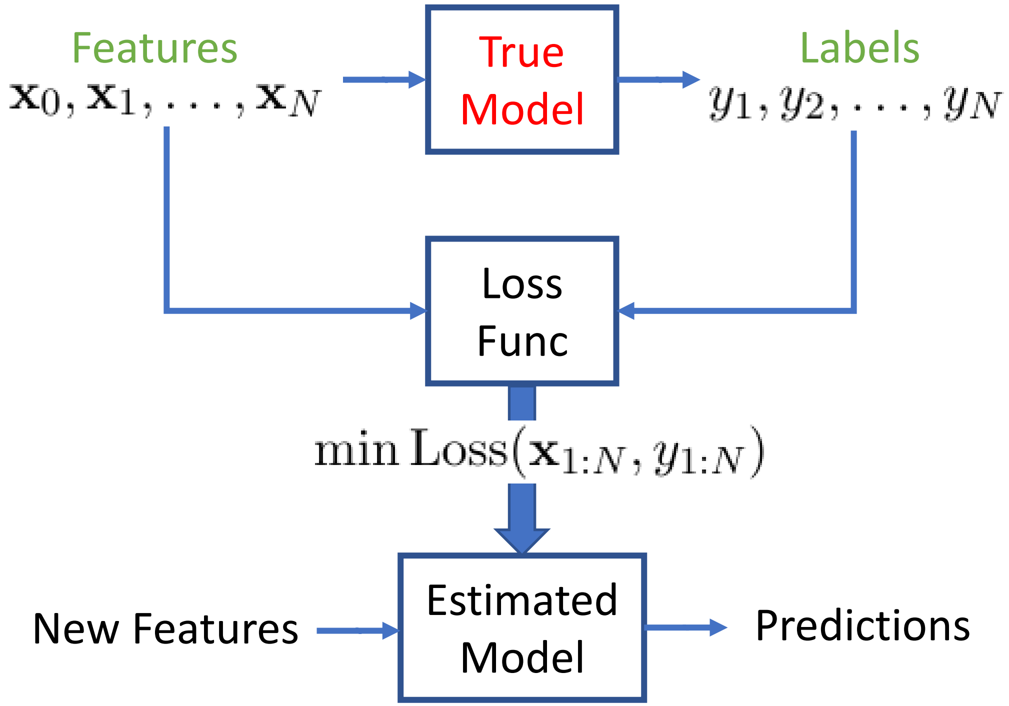


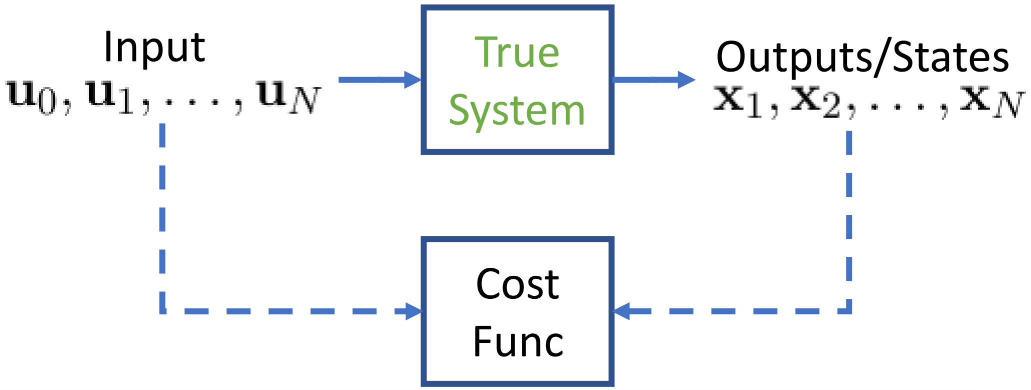




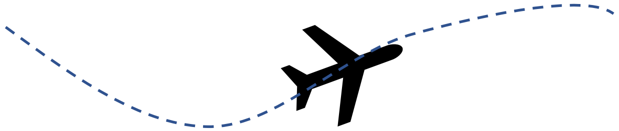


ML vs Control



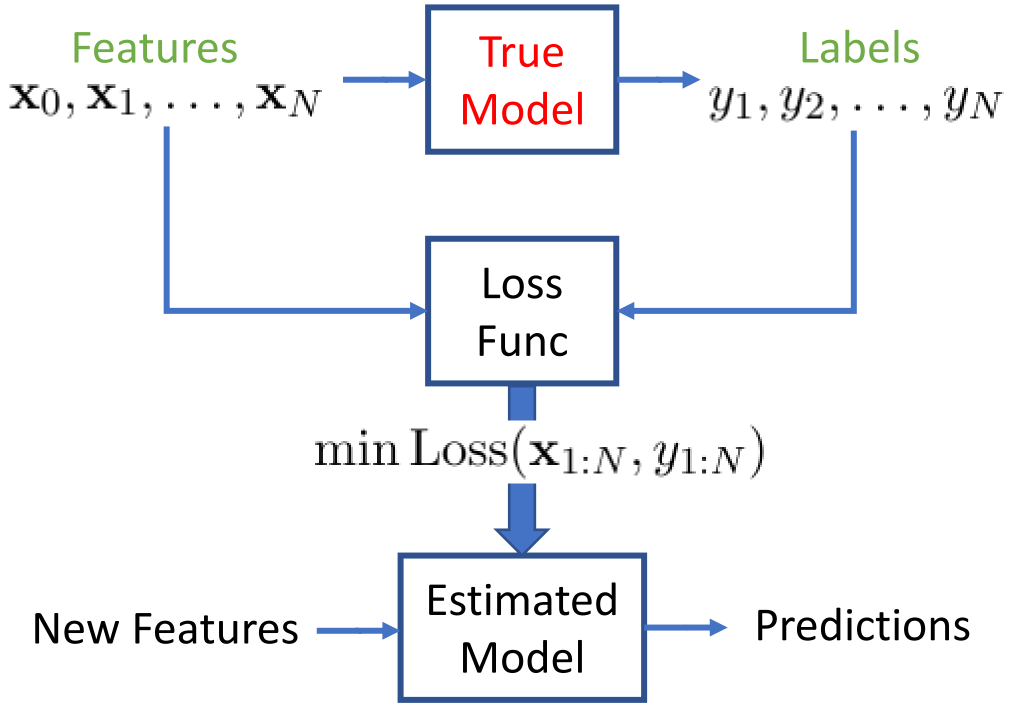


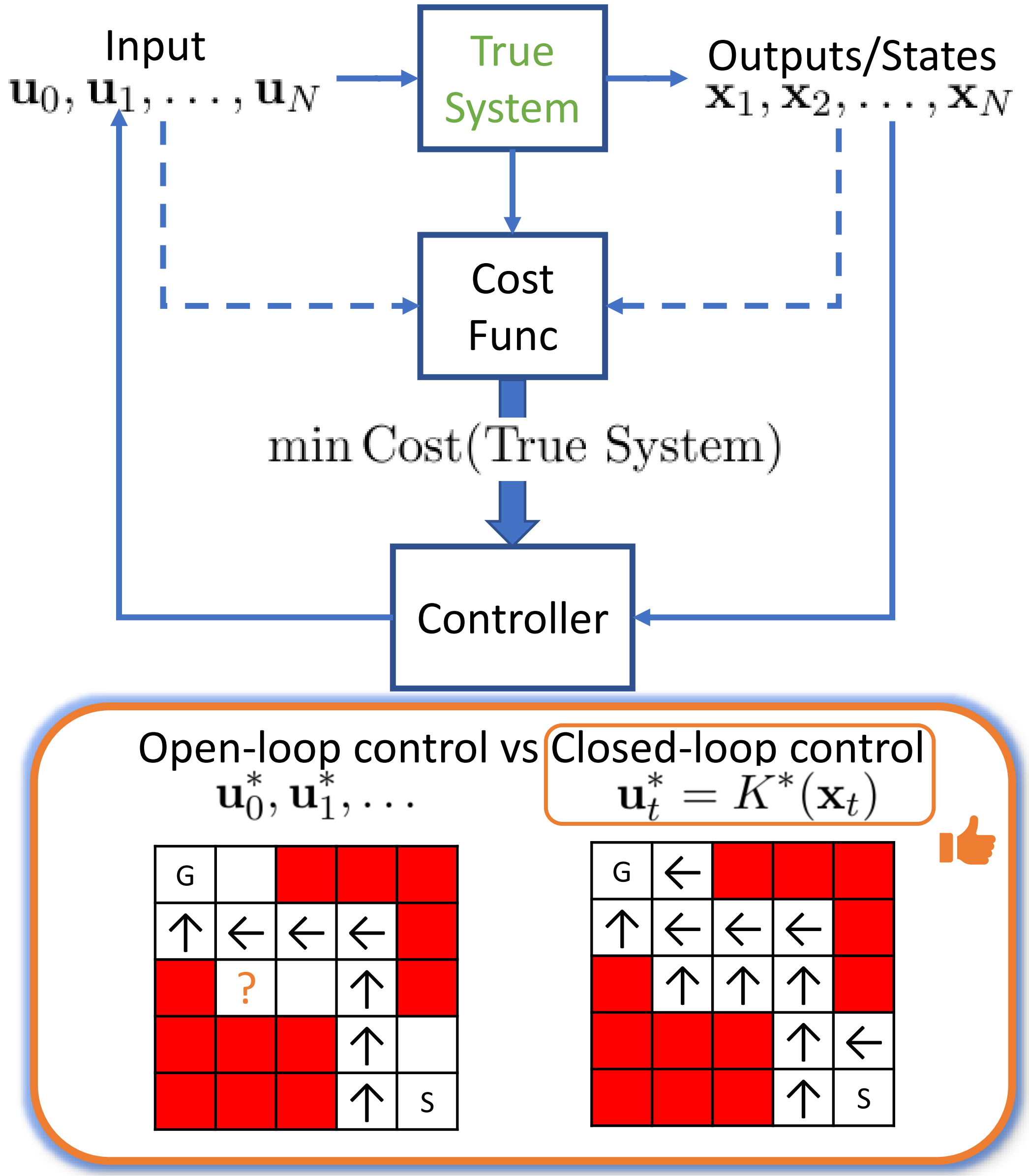
- Deviation from desired states/trajectories



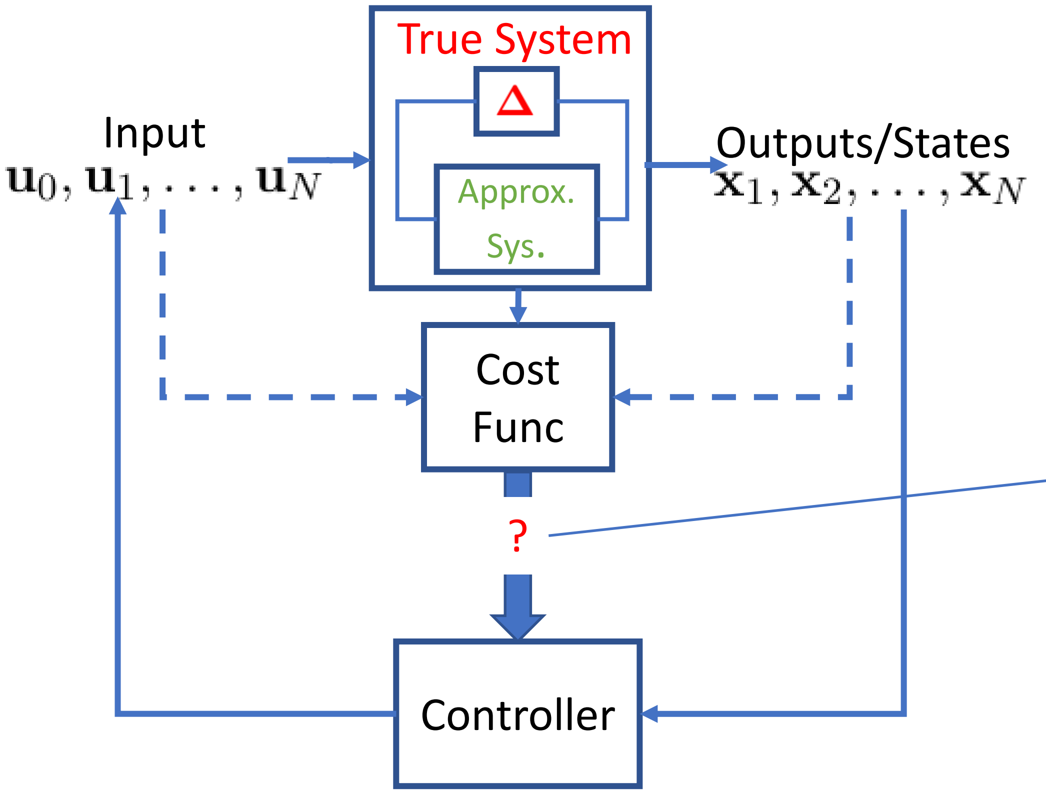
- Control efforts
- Safety guarantee
- Time limitation
- …

ML vs Control



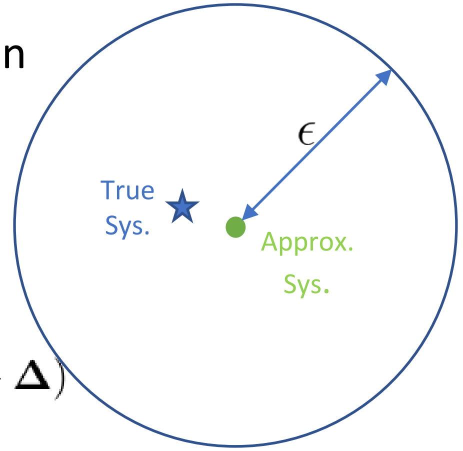


Robust Control vs Certainty Equivalent Control



Sources of $\boldsymbol{\Delta}$

- Estimation error in system identification
- Manufacturing variances
- Environmental changes



Robust Control

$$\min \max_{\|\boldsymbol{\Delta}\| \leq \epsilon} \text{Cost}(\text{Approx. Sys.} + \boldsymbol{\Delta})$$

Pro: robustness, Con: computation, suboptimality

Certainty Equivalent Control

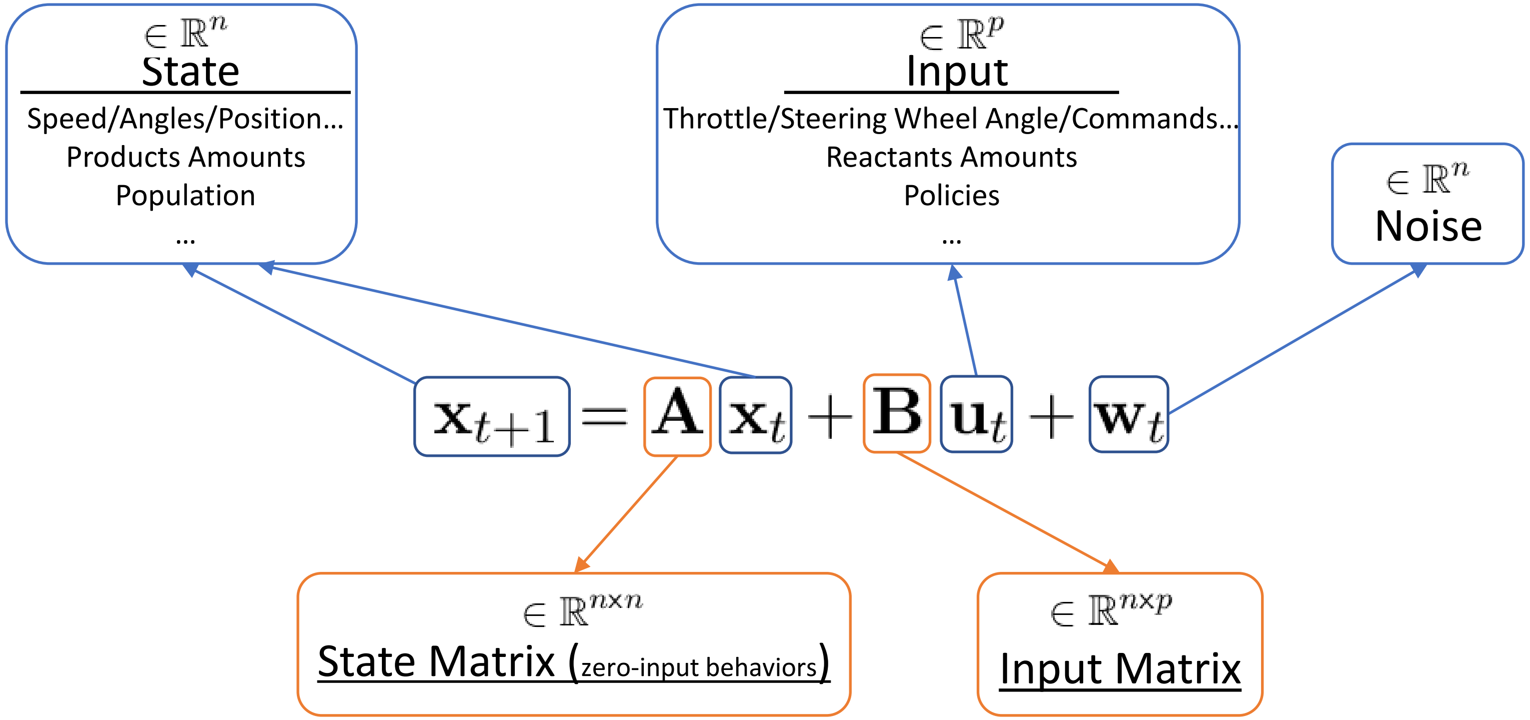
$$\min \text{Cost}(\text{Approx. Sys.})$$

Pro: computation, Con: robustness

Questions:

1. When does certainty equivalent control work?
2. Suboptimality compared w. the optimal controller?
3. Certainty equivalent control vs Robust control?

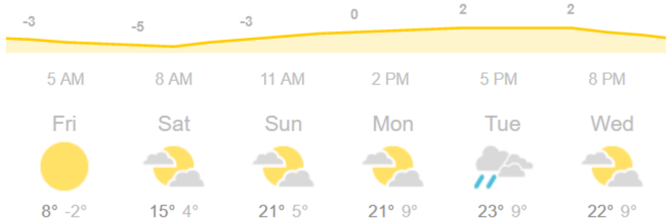
Linear Time-invariant Systems (LTI)

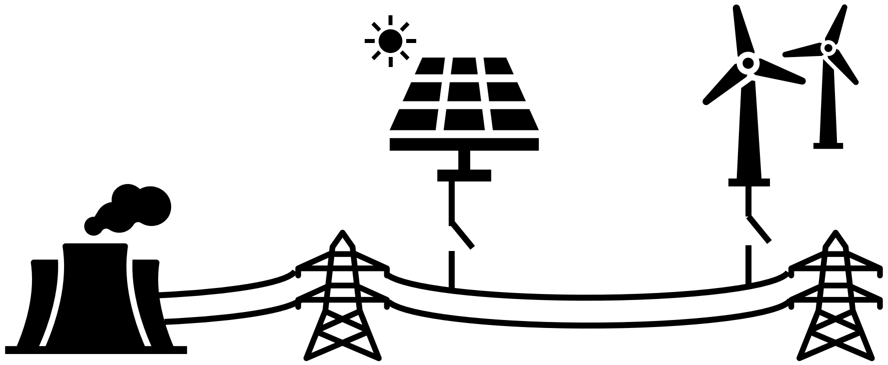


Linear Time-invariant Systems (LTI)

$$\mathbf{x}_{t+1} = \mathbf{A}\ \mathbf{x}_t + \mathbf{B}\ \mathbf{u}_t + \mathbf{w}_t$$

Is time-invariant dynamics accurate?





Markov Jump Systems (MJS)

$$\mathbf{x}_{t+1} = \mathbf{A}_{\omega(t)}\mathbf{x}_t + \mathbf{B}_{\omega(t)}\mathbf{u}_t + \mathbf{w}_t$$

$$\{\mathbf{A}_1, \mathbf{B}_1\}, \{\mathbf{A}_2, \mathbf{B}_2\}, \ldots \{\mathbf{A}_s, \mathbf{B}_s\}$$

s modes

Mode switching sequence

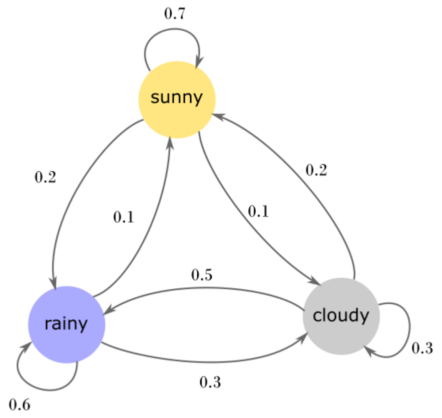
$$\omega(0), \omega(1), \cdots \sim MarkovChain(\mathbf{T})$$

$$\mathrm{P}(\underbrace{\omega(t+1) = j}_{\text{mode } j \text{ is active at time } t+1} \mid \underbrace{\omega(t) = i}_{\text{mode } i \text{ is active at time } t}) = \mathbf{T}_{ij}$$

mode j is active at time t+1

mode i is active at time t

- What is and why Markov chain?



- Given current mode, future modes do not depend on past modes
- Less memory, easier prediction and analysis
- Concepts and control strategies for LTI can be channeled to MJS via Markov chain

Markov Jump Systems (MJS)

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mode j is active at time $t+1$ — mode i is active at time t

- Relation to SGD
  - Data: $\{\mathbf{x}_{1:s}, y_{1:s}\}, y_i = \boldsymbol{\theta}_*^\top \mathbf{x}_i$
  - Loss: $\sum_{i=1}^s \frac{1}{2}(\boldsymbol{\theta}^\top \mathbf{x}_i - y_i)^2$
  - At time t,
    1. Pick $\omega(t) \in [s]$ w.p. $\frac{1}{s}$

Linear Quadratic Regulator (LQR) Problem

- Given cost matrices $\mathbf{Q}_{1:s}, \mathbf{R}_{1:s} \succ 0$, define quadratic cost

$$J_t := \mathbb{E}[\mathbf{x}_t^\intercal \mathbf{Q}_{\omega(t)} \mathbf{x}_t + \mathbf{u}_t^\intercal \mathbf{R}_{\omega(t)} \mathbf{u}_t]$$

Deviation from desired states (annotating $\mathbf{x}_t^\intercal \mathbf{Q}_{\omega(t)} \mathbf{x}_t$)

Control efforts (annotating $\mathbf{u}_t^\intercal \mathbf{R}_{\omega(t)} \mathbf{u}_t$)

and $J_{0:N} := \sum_{t=0}^{N} J_t$

Linear Quadratic Regulator (LQR) Problem

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- Finite-time horizon LQR $(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T}, \mathbf{Q}_{1:s}, \mathbf{R}_{1:s})$

$$\begin{aligned} \min_{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_N} \quad & J_{0:N} \\ \text{s.t. } \quad & \mathbf{x}_{t+1} = \mathbf{A}_{\omega(t)} \mathbf{x}_t + \mathbf{B}_{\omega(t)} \mathbf{u}_t + \mathbf{w}_t \\ & \mathrm{P}(\omega(t+1) = j \mid \omega(t) = i) = \mathbf{T}_{ij} \end{aligned}$$

Linear Quadratic Regulator (LQR) Problem

- Given cost matrices $\mathbf{Q}_{1:s}, \mathbf{R}_{1:s} \succ 0$, define quadratic cost

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- Finite-time horizon LQR $(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T}, \mathbf{Q}_{1:s}, \mathbf{R}_{1:s})$

$$\min_{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_N} \quad J_{0:N}$$
$$\text{s.t. } \mathbf{x}_{t+1} = \mathbf{A}_{\omega(t)} \mathbf{x}_t + \mathbf{B}_{\omega(t)} \mathbf{u}_t + \mathbf{w}_t$$
$$\mathrm{P}(\omega(t+1) = j \mid \omega(t) = i) = \mathbf{T}_{ij}$$

- Infinite-time horizon LQR $(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T}, \mathbf{Q}_{1:s}, \mathbf{R}_{1:s})$

$$\min_{\mathbf{u}_0, \mathbf{u}_1, \ldots} \quad \lim_{N \to \infty} \frac{1}{N} J_{0:N}$$
$$\text{s.t. } \mathbf{x}_{t+1} = \mathbf{A}_{\omega(t)} \mathbf{x}_t + \mathbf{B}_{\omega(t)} \mathbf{u}_t + \mathbf{w}_t$$
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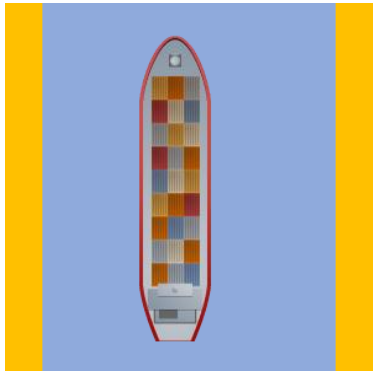


Optimal Solution for MJS-LQR

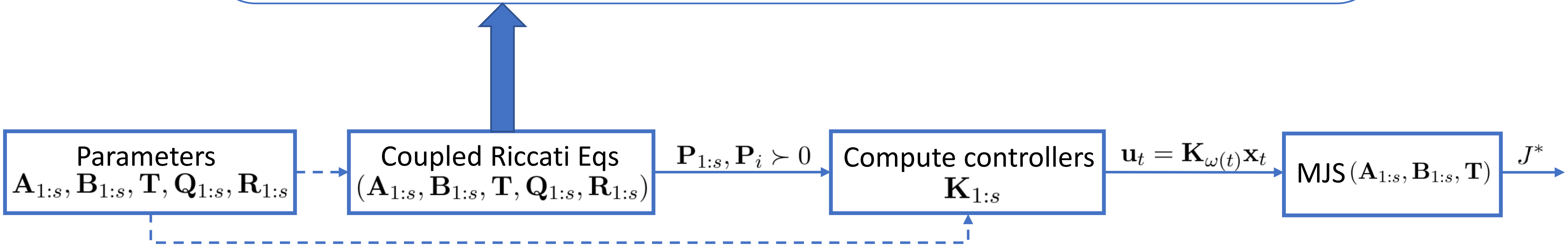
MJS-LQR $(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T}, \mathbf{Q}_{1:s}, \mathbf{R}_{1:s})$

$$\min_{\mathbf{u}_0, \mathbf{u}_1, \dots} \lim_{N \to \infty} \frac{1}{T} J_{0:N}$$

$$\text{s.t. } \mathbf{x}_{t+1} = \mathbf{A}_{\omega(t)} \mathbf{x}_t + \mathbf{B}_{\omega(t)} \mathbf{u}_t + \mathbf{w}_t$$

$$\mathrm{P}(\omega(t+1) = j \mid \omega(t) = i) = \mathbf{T}_{ij}$$

$$\begin{cases} \mathbf{P}_1 = \mathbf{A}_1^\intercal \left[\sum_{j=1}^s \mathbf{T}_{1j}\mathbf{P}_j\right] \mathbf{A}_1 + \mathbf{Q}_1 - \mathbf{A}_1^\intercal \left[\sum_{j=1}^s \mathbf{T}_{1j}\mathbf{P}_j\right] \mathbf{B}_1 \left(\mathbf{R}_1 + \mathbf{B}_1^\star \left[\sum_{j=1}^s \mathbf{T}_{1j}\mathbf{P}_j\right] \mathbf{B}_1\right)^{-1} \mathbf{B}_1^\intercal \left[\sum_{j=1}^s \mathbf{T}_{1j}\mathbf{P}_j\right] \mathbf{A}_1 \\ \vdots \\ \mathbf{P}_i = \mathbf{A}_i^\intercal \left[\sum_{j=1}^s \mathbf{T}_{ij}\mathbf{P}_j\right] \mathbf{A}_i + \mathbf{Q}_i - \mathbf{A}_i^\intercal \left[\sum_{j=1}^s \mathbf{T}_{ij}\mathbf{P}_j\right] \mathbf{B}_i \left(\mathbf{R}_i + \mathbf{B}_i^\star \left[\sum_{j=1}^s \mathbf{T}_{ij}\mathbf{P}_j\right] \mathbf{B}_i\right)^{-1} \mathbf{B}_i^\intercal \left[\sum_{j=1}^s \mathbf{T}_{ij}\mathbf{P}_j\right] \mathbf{A}_i \\ \vdots \\ \mathbf{P}_s = \mathbf{A}_s^\intercal \left[\sum_{j=1}^s \mathbf{T}_{sj}\mathbf{P}_j\right] \mathbf{A}_s + \mathbf{Q}_s - \mathbf{A}_s^\intercal \left[\sum_{j=1}^s \mathbf{T}_{sj}\mathbf{P}_j\right] \mathbf{B}_s \left(\mathbf{R}_s + \mathbf{B}_s^\star \left[\sum_{j=1}^s \mathbf{T}_{sj}\mathbf{P}_j\right] \mathbf{B}_s\right)^{-1} \mathbf{B}_s^\intercal \left[\sum_{j=1}^s \mathbf{T}_{sj}\mathbf{P}_j\right] \mathbf{A}_s \end{cases}$$



Optimal Solution for MJS-LQR

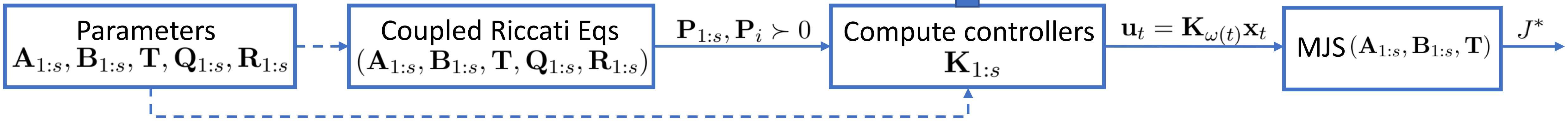
MJS-LQR $(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T}, \mathbf{Q}_{1:s}, \mathbf{R}_{1:s})$

$$\min_{\mathbf{u}_0, \mathbf{u}_1, \ldots} \lim_{N \to \infty} \frac{1}{T} J_{0:N}$$

$$\text{s.t. } \mathbf{x}_{t+1} = \mathbf{A}_{\omega(t)} \mathbf{x}_t + \mathbf{B}_{\omega(t)} \mathbf{u}_t + \mathbf{w}_t$$

$$\mathrm{P}(\omega(t+1) = j \mid \omega(t) = i) = \mathbf{T}_{ij}$$

$$\begin{cases} \mathbf{K}_1 = -\left(\mathbf{R}_1 + \mathbf{B}_1^\intercal \left[\sum_{j=1}^s \mathbf{T}_{1j} \mathbf{P}_j\right] \mathbf{B}_1\right)^{-1} \mathbf{B}_1^\intercal \left[\sum_{j=1}^s \mathbf{T}_{1j} \mathbf{P}_j\right] \mathbf{A}_1 \\ \vdots \\ \mathbf{K}_i = -\left(\mathbf{R}_i + \mathbf{B}_i^\intercal \left[\sum_{j=1}^s \mathbf{T}_{ij} \mathbf{P}_j\right] \mathbf{B}_i\right)^{-1} \mathbf{B}_i^\intercal \left[\sum_{j=1}^s \mathbf{T}_{ij} \mathbf{P}_j\right] \mathbf{A}_i \\ \vdots \\ \mathbf{K}_s = -\left(\mathbf{R}_s + \mathbf{B}_s^\intercal \left[\sum_{j=1}^s \mathbf{T}_{sj} \mathbf{P}_j\right] \mathbf{B}_s\right)^{-1} \mathbf{B}_s^\intercal \left[\sum_{j=1}^s \mathbf{T}_{sj} \mathbf{P}_j\right] \mathbf{A}_s \end{cases}$$

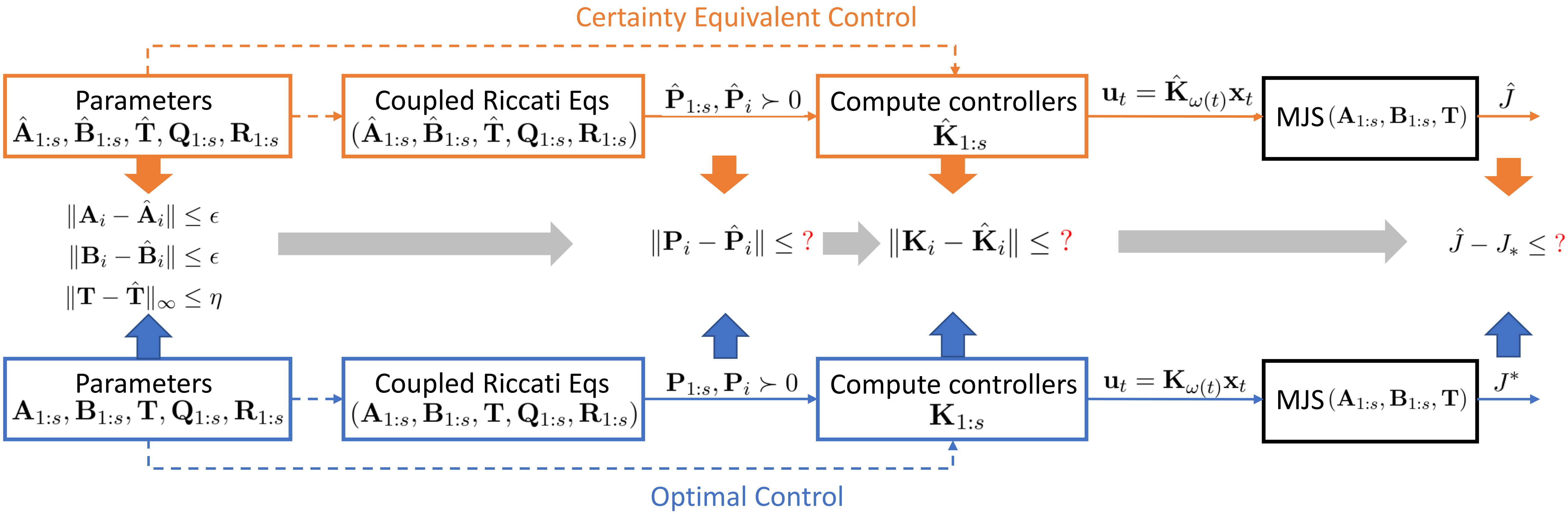


Certainty Equivalent Control for MJS-LQR

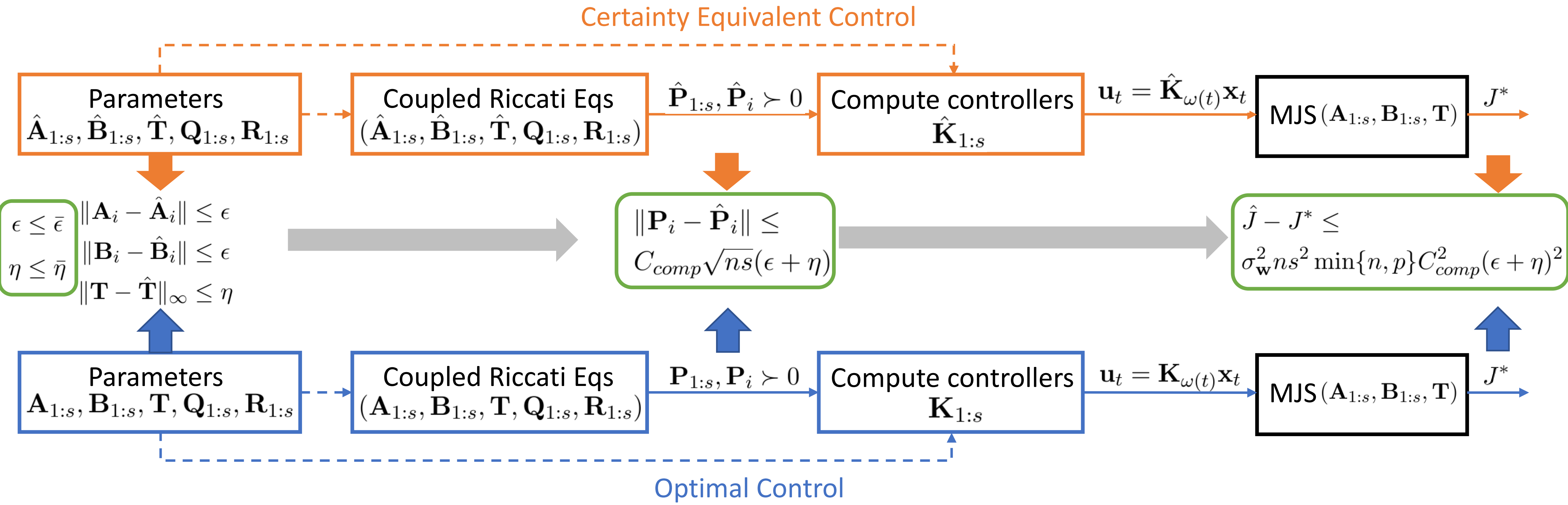
MJS-LQR $(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T}, \mathbf{Q}_{1:s}, \mathbf{R}_{1:s})$

$$\min_{\mathbf{u}_0,\mathbf{u}_1,\dots} \lim_{N\to\infty} \frac{1}{T} J_{0:N}$$
$$\text{s.t. } \mathbf{x}_{t+1} = \mathbf{A}_{\omega(t)}\mathbf{x}_t + \mathbf{B}_{\omega(t)}\mathbf{u}_t + \mathbf{w}_t$$
$$\mathrm{P}(\omega(t+1) = j \mid \omega(t) = i) = \mathbf{T}_{ij}$$

However, $\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T}$ are unknown, and we only know their *approximate* values $\hat{\mathbf{A}}_{1:s}, \hat{\mathbf{B}}_{1:s}, \hat{\mathbf{T}}$ such that $\|\mathbf{A}_i - \hat{\mathbf{A}}_i\| \leq \epsilon, \|\mathbf{B}_i - \hat{\mathbf{B}}_i\| \leq \epsilon, \|\mathbf{T} - \hat{\mathbf{T}}\|_\infty \leq \eta$

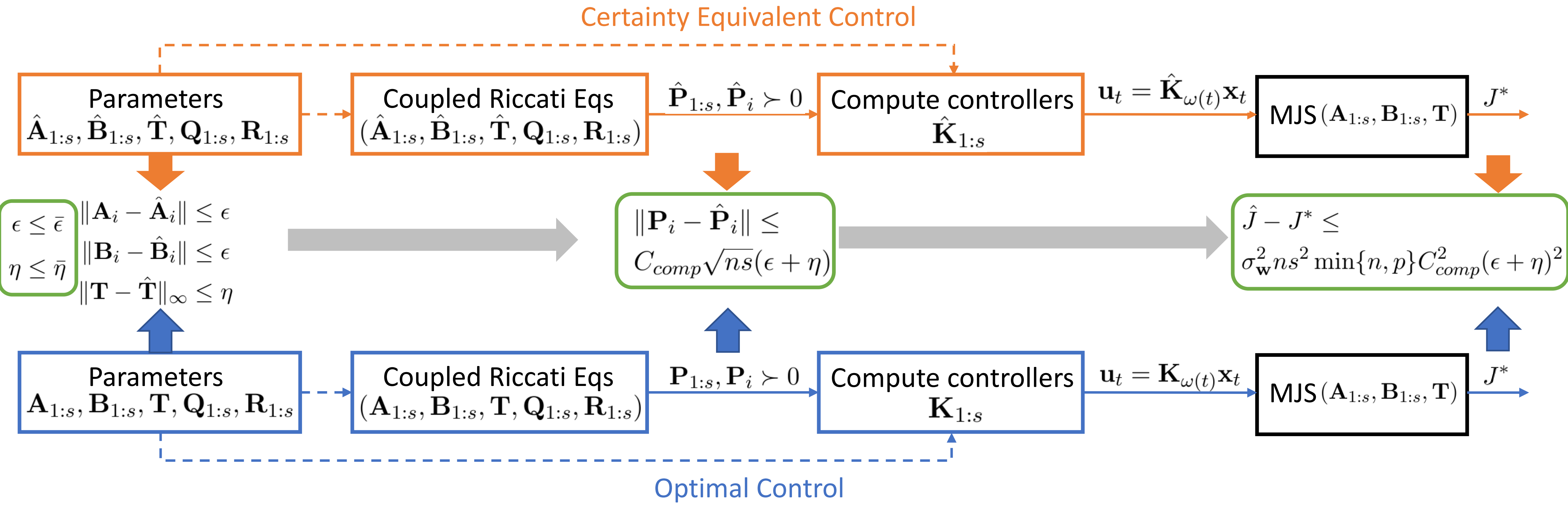


Results



$$C_{comp}, \bar{\epsilon}, \bar{\eta} \downarrow \text{ when stability} \uparrow \text{ and } \mathbf{R}_{1:s} \uparrow$$

Results

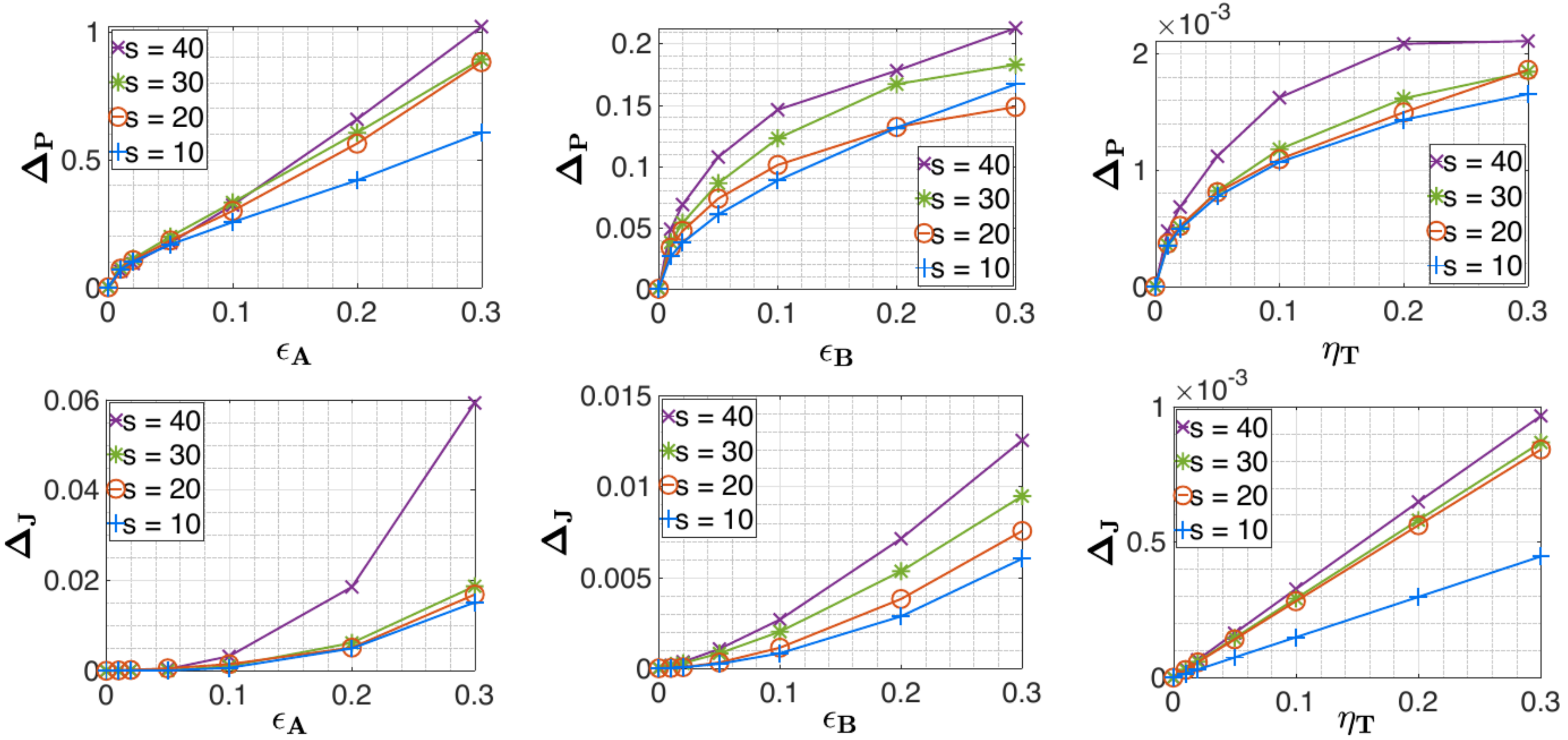


$s = 1$	$\bar{\epsilon}$	$\hat{J} - J^*$
Robust Control $\min \max_{\|\mathbf{\Delta}\| \leq \epsilon} \text{Cost}(\text{Approx. Sys.} + \mathbf{\Delta})$	Moderate	$\mathcal{O}(\epsilon)$
Certainty Equivalent Control $\min \text{Cost}(\text{Approx. Sys.})$	Stringent	$\mathcal{O}(\epsilon^2)$

Experiments

$$\|\mathbf{P}_i - \hat{\mathbf{P}}_i\| \leq C_{comp}\sqrt{ns}(\epsilon + \eta)$$

$$\hat{J} - J^* \leq \sigma_{\mathbf{w}}^2 n s^2 \min\{n, p\} C_{comp}^2 (\epsilon + \eta)^2$$



Future Work

- Offline Data-driven Control

$$\{\mathbf{u}_{0:M}, \mathbf{x}_{0:M}, \omega(0:M)\} \rightarrow \hat{\mathbf{A}}_{1:s}, \hat{\mathbf{B}}_{1:s}, \hat{\mathbf{T}} \rightarrow \hat{J}$$

- Online Adaptive Control

$$\{\mathbf{u}_{0:t}, \mathbf{x}_{0:t}, \omega(0:t)\} \rightarrow \hat{\mathbf{A}}_{1:s,t}, \hat{\mathbf{B}}_{1:s,t}, \hat{\mathbf{T}}_t \rightarrow \mathbf{u}_{t+1}, \mathbf{x}_{t+1}, \omega(t+1)$$

$$\rightarrow \hat{J}$$