# Certainty Equivalent Quadratic Control for Markov Jump Systems

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***Abstract*— Real-world control applications often involve complex dynamics subject to abrupt changes or variations. Markov jump linear systems (MJS) provide a rich framework for modeling such dynamics. Despite an extensive history, theoretical understanding of parameter sensitivities of MJS control is somewhat lacking. Motivated by this, we investigate robustness aspects of certainty equivalent model-based optimal control for MJS with a quadratic cost function. Given the uncertainty in the system matrices and in the Markov transition matrix is bounded by $\epsilon$ and $\eta$ respectively, robustness results are established for (i) the solution to coupled Riccati equations and (ii) the optimal cost, by providing explicit perturbation bounds that decay as $\mathcal{O}(\epsilon + \eta)$ and $\mathcal{O}((\epsilon + \eta)^2)$ respectively.**

## I. Introduction

The Linear Quadratic Regulator (LQR) is both theoretically well understood and commonly used in practice when the system dynamics are known. Its nice properties, e.g., admitting an elegant linear state feedback solution, make it a popular benchmark problem in reinforcement learning and adaptive control [1], [2], [3], [4], [5], [6], [7].

A natural generalization of linear time-invariant systems is Markov jump linear systems (MJS), which allow the dynamics of the underlying system to switch between multiple linear systems according to an underlying finite Markov chain. Similarly, a natural generalization of the LQR problem to MJS is to use mode-dependent cost matrices, which enables different control goals under different modes. While the optimal control for MJS-LQR is well understood when one has perfect knowledge of the system dynamics [8], [9], in practice we do not always know the exact system dynamics and the transition matrix. For instance, one might use system identification techniques to learn an approximate model for the system. Designing optimal controllers for MJS-LQR with this approximate system dynamics and transition matrix in place of the true ones leads to so-called certainty equivalent (CE) control which is used extensively in practice. However, a theoretical understanding of the suboptimality of the CE control for MJS-LQR is somewhat lacking. The main challenge here is the hybrid nature of the problem that requires consideration of both the system dynamics uncertainty $\epsilon$, and the underlying Markov transition matrix uncertainty $\eta$.

The solution of infinite horizon MJS-LQR involves coupled algebraic Riccati equations. Our goal is to understand the sensitivity of the solution of these equations and the corresponding optimal cost to perturbations in the system model. Toward this aim, we first establish an explicit $\mathcal{O}(\epsilon+\eta)$ perturbation bound for the solution to coupled algebraic Riccati equations that arise in the context of MJS-LQR. This in turn is used to establish an explicit $\mathcal{O}((\epsilon + \eta)^2)$ suboptimality bound on the cost. Finally, numerical experiments are provided to support our theoretical claims. Our proof strategy requires nontrivial advances over those of [4], [10]. Specifically, the coupled nature of these Riccati equations requires novel perturbation arguments, as they lack some of the nice properties of the standard Riccati equations, like uniqueness of solution under certain conditions or being amenable to matrix factorization based approaches.

*Related Work:* The performance analysis of CE control for the classical LQR problem for linear time invariant (LTI) systems relies on the perturbation/sensitivity analysis of the underlying algebraic Riccati equations (ARE), i.e. how much the ARE solution changes when the parameters in the equation are perturbed. This problem is studied in many works [11]. Early results on ARE solution perturbation bound are presented in [12] (continuous-time) and [10] (discrete-time). Most literature, however, only discusses perturbed solutions within the vicinity of the ground-truth solution. The uniqueness of such a perturbed solution is not discussed until [13], which is further refined in [14] to provide explicit perturbation bounds and generalization to complex equations. Tighter bounds are obtained [15] when the parameters have a special structure like sparsity.

Channelled by ARE perturbation results, the end-to-end CE LQR control suboptimality bound in terms of the dynamics perturbation is established in [4]. The field of CE MJS-LQR control and the corresponding coupled ARE (cARE) perturbation analysis, however, is not well studied. Two perturbation results [16], [17] for cARE only consider continuous-time cARE that arises in robust control applications and they are not directly applicable in MJS-LQR setting. Our work is also related to robust control for MJS (see, e.g., [18], [9]), where the focus is to numerically compute a controller to achieve a guaranteed cost under a given uncertainty bound. Whereas, we aim to characterize how the degradation in performance depends on perturbations in different parameters when CE control is used. Therefore, our work contributes to the body of work in robust control and CE control of MJS from a different perspective, and also paves the way to use these ideas in the context of learning-based adaptive control with performance guarantees as in our companion paper [19].

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## II. Preliminaries and Problem Setup

*Notations:* We use boldface uppercase (lowercase) letters to denote matrices (vectors). For a matrix $\mathbf{V}$, $\rho(\mathbf{V})$, $\underline{\sigma}(\mathbf{V})$, and $\|\mathbf{V}\|$ denote its spectral radius, smallest singular value, and spectral norm, respectively. We let $\|\mathbf{V}\|_+ := \|\mathbf{V}\| + 1$. $\mathbf{vec}(\mathbf{V})$ denotes the vectorization, and $\mathbf{V}_1 \otimes \mathbf{V}_2$ denotes the Kronecker product. $\mathbf{V}_{1:s}$ denotes a set of $s$ matrices $\{\mathbf{V}_i\}_{i=1}^s$ of same dimensions. We use $\mathbf{diag}(\mathbf{V}_{1:s})$ to denote a block diagonal matrix whose $i$-th diagonal block is given by $\mathbf{V}_i$. We define $[s] := \{1, 2, \dots, s\}$, $\underline{\sigma}(\mathbf{V}_{1:s}) := \min_{i\in[s]} \underline{\sigma}(\mathbf{V}_i)$, $\|\mathbf{V}_{1:s}\| := \max_{i\in[s]} \|\mathbf{V}_i\|$, and $\|\mathbf{V}_{1:s}\|_+ := \max_{i\in[s]} \|\mathbf{V}_i\|_+$. We use $\alpha\mathbf{U}_{1:s} + \beta\mathbf{V}_{1:s}$ to denote $\{\alpha\mathbf{U}_i + \beta\mathbf{V}_i\}_{i=1}^s$. Notation $\circ$ between two operators denotes the operator composition.

### A. Markov Jump Systems

We consider the problem of optimally controlling MJS, which are governed by the state equation,

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{A}_{\omega(t)}\mathbf{x}_t + \mathbf{B}_{\omega(t)}\mathbf{u}_t + \mathbf{w}_t \quad \text{s.t.} \\ \omega(t) &\sim \text{Markov Chain}(\mathbf{T}), \end{aligned} \tag{1}$$

where $\mathbf{x}_t \in \mathbb{R}^n$, $\mathbf{u}_t \in \mathbb{R}^p$ and $\mathbf{w}_t \in \mathbb{R}^n$ denote the state, input (or action) and noise at time $t$ respectively. Throughout, we assume $\mathbb{E}[\mathbf{x}_0\mathbf{x}_0^\intercal]$ is bounded, and $\{\mathbf{w}_t\}_{t=0}^\infty \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\mathbf{w}^2\mathbf{I}_n)$. There are $s$ modes in total, and the dynamics of the $i$-th mode is given by $(\mathbf{A}_i, \mathbf{B}_i)$. The active mode at time $t$ is indexed by $\omega(t) \in [s]$. In MJS the mode sequence $\{\omega(t)\}_{t=0}^\infty$ follows an ergodic Markov chain with transition matrix $\mathbf{T} \in \mathbb{R}_+^{s\times s}$ such that for all $t \geq 0$, the $ij$-th element of $\mathbf{T}$ denotes the conditional probability $[\mathbf{T}]_{ij} := \mathbb{P}\left(\omega(t+1) = j \mid \omega(t) = i\right), \forall i, j \in [s]$. Due to ergodicity, there exists a unique stationary distribution $\boldsymbol{\pi}_\infty \in \mathbb{R}^s$ such that $(\mathbf{T}^\intercal)^t\boldsymbol{\pi}_\infty \to \boldsymbol{\pi}_\infty$ as $t \to \infty$. Throughout, we assume the initial state $\mathbf{x}_0$, Markov chain $\{\omega(t)\}_{t=0}^\infty$, and noise $\{\mathbf{w}_t\}_{t=0}^\infty$ are mutually independent. We use $\text{MJS}(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T})$ to refer to an MJS parameterized by $(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T})$.

For the mode-dependent controller $\mathbf{K}_{1:s}$ that yields inputs $\mathbf{u}_t{=}\mathbf{K}_{\omega(t)}\mathbf{x}_t$, we use $\mathbf{L}_i{:=}\mathbf{A}_i + \mathbf{B}_i\mathbf{K}_i$ to denote the closed-loop state matrix for mode $i$. We use $\mathbf{x}_{t+1}{=}\mathbf{L}_{\omega(t)}\mathbf{x}_t$ to denote the noise-free closed-loop MJS . Due to the randomness in $\{\omega(t)\}_{t=0}^\infty$, it is common to consider the stability of MJS in the mean-square sense which is defined as follows.

**Definition 1.** *[9, Definitions 3.8, 3.40] (a) We say* MJS *in* (1) *with* $\mathbf{u}_t{=}0$ *is* **mean square stable (MSS)** *if there exists* $\mathbf{x}_\infty, \boldsymbol{\Sigma}_\infty$ *such that for any initial state/mode* $\mathbf{x}_0$, $\omega(0)$*, as* $t{\to}\infty$*, we have* $\|\mathbb{E}[\mathbf{x}_t] - \mathbf{x}_\infty\|{\to}0$ *and* $\|\mathbb{E}[\mathbf{x}_t\mathbf{x}_t^\intercal] - \boldsymbol{\Sigma}_\infty\|{\to}0$*. In the noise-free case* ($\mathbf{w}_t{=}0$)*, we have* $\mathbf{x}_\infty{=}0$, $\boldsymbol{\Sigma}_\infty{=}0$*. (b) We say* MJS *in* (1) *with* $\mathbf{w}_t{=}0$ *is* **(mean square) stabilizable** *if there exists mode-dependent controller* $\mathbf{K}_{1:s}$ *such that the closed-loop* MJS $\mathbf{x}_{t+1}{=}(\mathbf{A}_{\omega(t)}{+}\mathbf{B}_{\omega(t)}\mathbf{K}_{\omega(t)})\mathbf{x}_t$ *is* MSS*. We call such* $\mathbf{K}_{1:s}$ *a stabilizing controller.*

One can check the stabilizability of an MJS via linear matrix inequalities [9, Proposition 3.42]. It is well-known that the stability of non-switching systems is related to the spectral radius of the state matrix. Similarly, the mean-square stability of an autonomous MJS $\mathbf{x}_{t+1} = \mathbf{L}_{\omega(t)}\mathbf{x}_t$ is related to the spectral radius of the augmented state matrix: $\tilde{\mathbf{L}} \in \mathbb{R}^{sn^2 \times sn^2}$ with $ij$-th $n^2{\times}n^2$ block given by

$$[\tilde{\mathbf{L}}]_{ij} := [\mathbf{T}]_{ij}\mathbf{L}_i^\intercal \otimes \mathbf{L}_i^\intercal, \quad \forall\, i, j \in [s]. \tag{2}$$

Define the operator, $\varphi_i(\mathbf{V}_{1:s}) := \sum_{j=1}^s [\mathbf{T}]_{ij}\mathbf{V}_j$ for all $i \in [s]$, then we have the following results regarding the MSS.

**Lemma 2.** *[9, Theorem 3.9] The following are equivalent: (a)* MJS $\mathbf{x}_{t+1}{=}\mathbf{L}_{\omega(t)}\mathbf{x}_t$ *is* MSS*; (b)* $\rho(\tilde{\mathbf{L}}){<}1$*; (c) there exists* $\mathbf{V}_{1:s}$ *with* $\mathbf{V}_i{\succ}0$*, such that* $\mathbf{V}_i{-}\mathbf{L}_i^\intercal\varphi_i(\mathbf{V}_{1:s})\mathbf{L}_i{\succ}0$, $\forall i \in [s]$.

These assertions reduce to the classical stability results regarding spectral radius and Lyapunov equation when $s = 1$. Moreover, it can be shown that the augmented matrix $\tilde{\mathbf{L}}^\intercal$ maps $\{\mathbb{E}[\mathbf{x}_t\mathbf{x}_t^\intercal\mathbf{1}_{\omega(t)=i}]\}_{i=1}^s$ to $\{\mathbb{E}[\mathbf{x}_{t+1}\mathbf{x}_{t+1}^\intercal\mathbf{1}_{\omega(t+1)=i}]\}_{i=1}^s$ [9, p.35], hence its spectral radius determines MSS.

### B. Linear Quadratic Regulator

The optimal control problem we consider in this paper is the following Markov jump system infinite-horizon linear quadratic regulator (MJS-LQR) problem where we seek to minimize the long-term average quadratic cost $J(\mathbf{u}_0, \mathbf{u}_1, \dots) := \limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^T \mathbf{x}_t^\intercal\mathbf{Q}_{\omega(t)}\mathbf{x}_t{+}\mathbf{u}_t^\intercal\mathbf{R}_{\omega(t)}\mathbf{u}_t\right]$, i.e.

$$\begin{aligned} &\inf \quad J(\mathbf{u}_0, \mathbf{u}_1, \dots) \\ &\text{s.t.} \quad \mathbf{x}_t, \omega(t) \sim \text{MJS}(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T}). \end{aligned} \tag{3}$$

Matrices $\mathbf{Q}_{\omega(t)}$ and $\mathbf{R}_{\omega(t)}$ are mode-dependent cost matrices chosen by users, and the expectation is over the randomness of initial state $\mathbf{x}_0$, noise $\{\mathbf{w}_t\}_{t=0}^\infty$ and Markovian modes $\{\omega(t)\}_{t=0}^\infty$. Unlike classical LQR for LTI systems, where cost matrices are usually fixed throughout the time horizon, the mode-dependent cost matrices in MJS-LQR allows us to have different control goals under different modes. In this work, we are interested in the state feedback solution under the mode-dependent controller, which is guaranteed to exist under the following assumption.

**Assumption 3.** *(a) For all* $i \in [s]$, $\mathbf{Q}_i \succ 0$ *and* $\mathbf{R}_i \succ 0$*; (b) the* MJS *in* (1) *with* $\mathbf{w}_t = 0$ *is stabilizable.*

Similar to the algebraic Riccati equation for LTI-LQR, the optimal solution to (3) is closely related to the following $s$ coupled Riccati equations: for $i = 1, 2, \cdots, s$,

$$\begin{aligned} \mathbf{P}_i = \mathbf{A}_i^\intercal\varphi_i(\mathbf{P}_{1:s})\mathbf{A}_i + \mathbf{Q}_i - \mathbf{A}_i^\intercal\varphi_i(\mathbf{P}_{1:s})\mathbf{B}_i \\ \cdot \left(\mathbf{R}_i + \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}_{1:s})\mathbf{B}_i\right)^{-1}\mathbf{B}_i^\intercal\varphi_i(\mathbf{P}_{1:s})\mathbf{A}_i \end{aligned} \tag{4}$$

with $\mathbf{P}_{1:s}$ as unknowns. We refer (4) as coupled discrete-time algebraic Riccati equations (cDARE), and use notation $\text{cDARE}(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T})$ to denote the parametrized form, where the Markov transition matrix $\mathbf{T}$ determines the operator $\varphi$ in (4). In practice, cDARE can be solved efficiently either with LMIs or via value iteration [9]. We know the following about the solution to (3) and (4).

**Lemma 4.** *[9, Theorem 4.6 and Corollary A.21] Under Assumption 3,* $\text{cDARE}(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T})$ *has a unique solution* $\mathbf{P}_{1:s}^\star$ *among* $\{\mathbf{P}_{1:s} : \mathbf{P}_i \succeq 0,\ \forall\, i\}$*, and* $\mathbf{P}_i^\star \succ 0$ *for all* $i \in [s]$*. Moreover, the controller* $\mathbf{K}_{1:s}^\star$ *with*

$$\mathbf{K}_i^\star = -\left(\mathbf{R}_i + \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}_{1:s}^\star)\mathbf{B}_i\right)^{-1}\mathbf{B}_i^\intercal\varphi_i(\mathbf{P}_{1:s}^\star)\mathbf{A}_i \tag{5}$$

*stabilizes* MJS *in* (1) *and minimizes the cost* (3) *with input* $\mathbf{u}_t{=}\mathbf{K}^\star_{\omega(t)}\mathbf{x}_t$ *and optimal cost* $J^\star{=}\sigma_{\mathbf{w}}^2\mathbf{tr}(\sum_{i\in[s]}\boldsymbol{\pi}_\infty(i)\mathbf{P}^\star_i)$.

### C. Certainty Equivalent Controller

In this work we seek to control MJS$(\mathbf{A}_{1:s},\mathbf{B}_{1:s},\mathbf{T})$ with unknown dynamics $(\mathbf{A}_{1:s},\mathbf{B}_{1:s},\mathbf{T})$ based on approximate parameters $(\hat{\mathbf{A}}_{1:s},\hat{\mathbf{B}}_{1:s},\hat{\mathbf{T}})$ that satisfy

$$\|\mathbf{A}_i-\hat{\mathbf{A}}_i\|\leq\epsilon,\ \|\mathbf{B}_i-\hat{\mathbf{B}}_i\|\leq\epsilon,\ \|\mathbf{T}-\hat{\mathbf{T}}\|_\infty\leq\eta. \tag{6}$$

The cost matrices $(\mathbf{Q}_{1:s},\mathbf{R}_{1:s})$ are assumed known and the modes $\{\omega(t)\}_{t=0}^\infty$ are observed at run-time. We analyze the CE approach, that is, using the approximate parameters $(\hat{\mathbf{A}}_{1:s},\hat{\mathbf{B}}_{1:s},\hat{\mathbf{T}})$, we solve the perturbed cDARE$(\hat{\mathbf{A}}_{1:s},\hat{\mathbf{B}}_{1:s},\hat{\mathbf{T}})$,

$$\begin{aligned}\mathbf{P}_i &= \hat{\mathbf{A}}_i^\intercal\hat{\varphi}_i(\mathbf{P}_{1:s})\hat{\mathbf{A}}_i+\mathbf{Q}_i-\hat{\mathbf{A}}_i^\intercal\hat{\varphi}_i(\mathbf{P}_{1:s})\hat{\mathbf{B}}_i\\ &\quad\cdot\big(\mathbf{R}_i+\hat{\mathbf{B}}_i^\intercal\hat{\varphi}_i(\mathbf{P}_{1:s})\hat{\mathbf{B}}_i\big)^{-1}\hat{\mathbf{B}}_i^\intercal\hat{\varphi}_i(\mathbf{P}_{1:s})\hat{\mathbf{A}}_i, \end{aligned}\tag{7}$$

for all $i\in[s]$ and $\mathbf{P}_i\succeq 0$, where the operator $\hat{\varphi}$ is defined as $\hat{\varphi}_i(\mathbf{V}_{1:s}):=\sum_{j=1}^s[\hat{\mathbf{T}}]_{ij}\mathbf{V}_j$. Let $\hat{\mathbf{P}}_{1:s}$ be the positive definite solution of (7), then the CE controller is given by $\hat{\mathbf{K}}_{1:s}$ with

$$\hat{\mathbf{K}}_i=-\big(\mathbf{R}_i+\hat{\mathbf{B}}_i^\intercal\hat{\varphi}_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{B}}_i\big)^{-1}\hat{\mathbf{B}}_i^\intercal\hat{\varphi}_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{A}}_i. \tag{8}$$

Lastly, we apply the input $\hat{\mathbf{u}}_t=\hat{\mathbf{K}}_{\omega(t)}\mathbf{x}_t$ to control the true MJS$(\mathbf{A}_{1:s},\mathbf{B}_{1:s},\mathbf{T})$.

Let $\hat{J}$ denote the cost incurred by playing the CE controller $\hat{\mathbf{K}}_{1:s}$. In the next section, we address the following questions: (a) When can the perturbed cDARE in (7) be guaranteed to have a unique positive semi-definite solution $\hat{\mathbf{P}}_{1:s}$? (b) What is a tight upper bound on $\|\hat{\mathbf{P}}_{1:s}-\mathbf{P}^\star_{1:s}\|$? (c) When does $\hat{\mathbf{K}}_{1:s}$ stabilize the true MJS? (d) How large is the suboptimality gap $\hat{J}-J^\star$?

## III. Perturbation Analysis for MJS-LQR

We first introduce a few more concepts and notations. We use $\mathbf{L}^\star_i:=\mathbf{A}_i+\mathbf{B}_i\mathbf{K}^\star_i$ to denote the closed-loop state matrix under the optimal MJS-LQR controller (5), and define the augmented state matrix $\tilde{\mathbf{L}}^\star$ similar to (2) such that its $ij$-th block is given by $[\tilde{\mathbf{L}}^\star]_{ij}:=[\mathbf{T}]_{ij}\mathbf{L}_i^{\star\intercal}\otimes\mathbf{L}_i^{\star\intercal}$. From Lemma 4, we know the closed-loop MJS $\mathbf{x}_{t+1}=\mathbf{L}^\star_{\omega(t)}\mathbf{x}_t$ is MSS, thus $\rho(\tilde{\mathbf{L}}^\star)<1$ by Lemma 2. We let $\rho^\star:=\rho(\tilde{\mathbf{L}}^\star)$ and define the following to quantify the decay of $\tilde{\mathbf{L}}^\star$.

$$\tau^\star:=\sup_{k\in\mathbb{N}}\|(\tilde{\mathbf{L}}^\star)^k\|/\rho^{\star k}. \tag{9}$$

Note that $\tau^\star$ is finite by Gelfand's formula, and by definition we have $\tau^\star\geq 1$. $\tau^\star$ measures the transient response of a non-switching system with state matrix $\tilde{\mathbf{L}}^\star$ and can be upper bounded by its $\mathcal{H}_\infty$ norm [20].

To the ease of exposition, we define a few constants:

$$\begin{aligned}\xi &:= \min\{\|\mathbf{B}_{1:s}\|_+^{-2}\|\mathbf{R}_{1:s}^{-1}\|_+^{-1}\|\mathbf{L}^\star_{1:s}\|_+^{-2},\underline{\sigma}(\mathbf{P}^\star_{1:s})\},\\ C_\epsilon &:= 6\|\mathbf{A}_{1:s}\|_+^2\|\mathbf{B}_{1:s}\|_+\|\mathbf{P}^\star_{1:s}\|_+^2\|\mathbf{R}_{1:s}^{-1}\|_+,\\ C_\epsilon^u &:= 6C_\epsilon^{-1}\|\mathbf{B}_{1:s}\|_+^{-2}\|\mathbf{P}^\star_{1:s}\|_+^{-1}\|\mathbf{R}_{1:s}^{-1}\|_+^{-1},\\ C_\eta &:= 2\|\mathbf{A}_{1:s}\|_+^2\|\mathbf{B}_{1:s}\|_+^4\|\mathbf{P}^\star_{1:s}\|_+^3\|\mathbf{R}_{1:s}^{-1}\|_+^2,\\ C_\eta^u &:= 6C_\eta^{-1},\\ \Gamma_\star &:= \max\{\|\mathbf{A}_{1:s}\|_+,\|\mathbf{B}_{1:s}\|_+,\|\mathbf{P}^\star_{1:s}\|_+,\|\mathbf{K}^\star_{1:s}\|_+\}.\\ \bar{\epsilon}_\mathbf{K} &:= \frac{1-\rho^\star}{2\sqrt{s}\tau^\star(1+2\|\mathbf{L}^\star_{1:s}\|)\|\mathbf{B}_{1:s}\|}\end{aligned}\tag{10}$$

In the following, we will show that despite being coupled, cDARE for MJS-LQR satisfies nice local Lipschitz properties. To be more precise, we show that if the approximate MJS is accurate enough, i.e., $\epsilon$ and $\eta$ are sufficiently small, we can guarantee that, not only the positive definite solution $\hat{\mathbf{P}}_{1:s}$ to the perturbed cDARE uniquely exists, but also $\hat{\mathbf{P}}_{1:s}$ is guaranteed to be close to $\mathbf{P}^\star_{1:s}$.

**Theorem 5.** *Under Assumption 3, and as long as* $\epsilon\leq\min\Big\{\frac{C_\epsilon^u\xi(1-\rho^\star)^2}{204ns\tau^{\star 2}},\ \|\mathbf{B}_{1:s}\|\Big\}$, $\eta\leq\frac{C_\eta^u\xi(1-\rho^\star)^2}{48ns\tau^{\star 2}}$, *the perturbed* cDARE *in* (7) *is guaranteed to have a unique solution* $\hat{\mathbf{P}}_{1:s}$ *in* $\{\mathbf{X}_{1:s}:\mathbf{X}_i\succeq 0,\ \forall i\}$ *such that* $\hat{\mathbf{P}}_i\succ 0$ *for all* $i$ *and*

$$\|\hat{\mathbf{P}}_{1:s}-\mathbf{P}^\star_{1:s}\|\leq\frac{\sqrt{ns}\tau^\star}{1-\rho^\star}(C_\epsilon\epsilon+C_\eta\eta). \tag{11}$$

From the constants, we see we would have milder requirements on $\epsilon$ and $\eta$ and a tighter bound on $\|\hat{\mathbf{P}}_{1:s}-\mathbf{P}^\star_{1:s}\|$ when (i) $\|\mathbf{A}_{1:s}\|$, $\|\mathbf{B}_{1:s}\|$, (ii) $\|\mathbf{L}^\star_{1:s}\|$, $\tau^\star$, and (iii) $\|\mathbf{R}_{1:s}^{-1}\|$ are smaller. These translate to the cases when (i) the true MJS is easier to stabilize; (ii) the closed-loop MJS under the optimal controller is more stable; and (iii) the input dominates more in the cost function. The role of $\tau^\star$ in this theorem is closely related to the damping property in ARE perturbation analysis [12]. The coefficients for $\epsilon$ and $\eta$ on the RHS of (11) are also known as condition numbers in algebraic Riccati equation sensitivity literature [14].

Note that the perturbation upper bound in Theorem 5, when setting $s=1$ and $\eta=0$, is consistent with [4, Proposition 1] developed for the LTI case except that we suffer an additional $\sqrt{n}$ term. This is because, due to the coupled nature of $\hat{\mathbf{P}}_{1:s}$ through cDARE, we proceed by first vectorizing and stacking cDARE into a single equation to evaluate $[\mathbf{vec}(\hat{\mathbf{P}}_1)^\intercal,\cdots,\mathbf{vec}(\hat{\mathbf{P}}_s)^\intercal]^\intercal$, then convert it back to $\hat{\mathbf{P}}_{1:s}$ through reshaping. Certain norm equivalency arguments (Fact 2) are needed to carry perturbation results through this back-and-forth reshaping steps, which produces this additional $\sqrt{n}$. On the other hand, these steps and thus the $\sqrt{n}$ term are not needed for the LTI case, since only a single Riccati equation is involved.

It is easy to extend this result to the cases when an approximate $\hat{\mathbf{Q}}_{1:s}$ with $\|\hat{\mathbf{Q}}_{1:s}{-}\mathbf{Q}_{1:s}\|{\leq}\epsilon$ is used in place of $\mathbf{Q}_{1:s}$ in the computations, which can be useful when the cost includes a term of the form $\|\mathbf{y}_t\|^2$ where $\mathbf{y}_t{=}\mathbf{C}_{\omega(t)}\mathbf{x}_t$ represents the output, and we only have an approximate parameter $\hat{\mathbf{C}}_{1:s}$. In this case, $\mathbf{Q}_i{=}\mathbf{C}_i^\intercal\mathbf{C}_i$ and $\hat{\mathbf{Q}}_i{=}\hat{\mathbf{C}}_i^\intercal\hat{\mathbf{C}}_i$.

In the next result, we leverage Theorem 5 to show how the controller $\hat{\mathbf{K}}_{1:s}$ computed from a perturbed cDARE solution deviates from the optimal one, i.e., how $\|\hat{\mathbf{K}}_{1:s}{-}\mathbf{K}^\star_{1:s}\|$ depends on $\epsilon$ and $\eta$, and when $\hat{\mathbf{K}}_{1:s}$ stabilizes the true MJS (such that $\hat{J}$ will be bounded). Moreover, with the help of [21, Lemma 3], which provides a relation between suboptimality gap $\hat{J}{-}J^\star$ and $\|\hat{\mathbf{K}}_{1:s}{-}\mathbf{K}^\star_{1:s}\|$, we establish an upper bound for $\hat{J}{-}J^\star$ in terms of $\epsilon$ and $\eta$.

**Theorem 6.** *Under Assumptions 3, suppose* $\epsilon$ *and* $\eta$ *satisfy the bounds in Theorem 5 and* $C_\epsilon\epsilon+C_\eta\eta\leq\frac{(1-\rho^\star)\min\{\Gamma_\star,\underline{\sigma}(\mathbf{R}_{1:s})^2\bar{\epsilon}_\mathbf{K}\}}{28\sqrt{ns}\tau^\star\Gamma_\star^3(\underline{\sigma}(\mathbf{R}_{1:s})+\Gamma_\star^3)}$. *Then CE controller* $\hat{\mathbf{K}}_{1:s}$ *stabilizes* *the true MJS and*

$$\|\mathbf{K}^\star_{1:s} - \hat{\mathbf{K}}_{1:s}\| \le 28\sqrt{ns}\tau^\star\Gamma_\star^3 \frac{(\underline{\sigma}(\mathbf{R}_{1:s}) + \Gamma_\star^3)}{(1-\rho^\star)\underline{\sigma}(\mathbf{R}_{1:s})^2}(C_\epsilon \epsilon + C_\eta \eta) \quad (12)$$

$$\hat{J} - J^\star \le 1600\sigma_{\mathbf{w}}^2 \frac{s^{2.5} n^{1.5} \min\{n,p\} \tau^{\star 3} \Gamma_\star^6}{(1-\rho^\star)^3} \cdot \frac{(\|\mathbf{R}_{1:s}\| + \Gamma_\star^3)^3}{\underline{\sigma}(\mathbf{R}_{1:s})^4}(C_\epsilon \epsilon + C_\eta \eta)^2. \quad (13)$$

This result states that the suboptimality has quadratic dependency on the uncertainties $\epsilon$ and $\eta$, and degrades when the MJS has larger number of modes $s$, system order $n$, or noise variance $\sigma_{\mathbf{w}}^2$. Similar to the earlier discussion, Theorem 6 is also consistent with its LTI counterpart [4, Theorem 1] except the $n$ term.

Our sub-optimality result has important implications in data-driven control for MJS. Suppose the uncertainties $\epsilon$ and $\eta$ in the system dynamics and the transition matrix are due to estimation errors induced by a system identification procedure that uses $T$ samples. Then, if the estimation error decays as $\mathcal{O}(1/\sqrt{T})$ (which is typical for $\epsilon$ as in learning LTI [22], [23] and for $\eta$ in learning Markov chains [24]), Theorem 6 implies that the suboptimality decays as $\mathcal{O}(1/T)$. Thus, given a desired sub-optimality level for the CE controller, one can use this relation to infer the required number of samples, which has been employed in our companion paper [19] to establish regret analysis for adaptive control.

## IV. Numerical Experiments

In this section, we present some numerical results to support our proposed theory. We fix $n{=}10$ and $p{=}5$. The true system matrices $(\mathbf{A}_{1:s}, \mathbf{B}_{1:s})$ were generated randomly from the standard normal distribution. We scaled each $\mathbf{A}_i$ to have spectral radius equal to 0.3 to obtain a mean square stable MJS. We set $\mathbf{Q}_i{=}\underline{\mathbf{Q}}_i\underline{\mathbf{Q}}_i^\top$, $\mathbf{R}_i{=}\underline{\mathbf{R}}_i\underline{\mathbf{R}}_i^\top$, $\hat{\mathbf{A}}_i{=}\mathbf{A}_i{+}\epsilon_{\mathbf{A}}\underline{\mathbf{A}}_i$, and $\hat{\mathbf{B}}_i{=}\mathbf{B}_i{+}\epsilon_{\mathbf{B}}\underline{\mathbf{B}}_i$, where $\underline{\mathbf{Q}}_i$, $\underline{\mathbf{R}}_i$, $\underline{\mathbf{A}}_i$, and $\underline{\mathbf{B}}_i$ were generated randomly from the standard normal distribution; and $\epsilon_{\mathbf{A}}$ and $\epsilon_{\mathbf{B}}$ are some fixed scalars. Here we experimentally study the influences of perturbation on $\mathbf{A}_{1:s}$ and $\mathbf{B}_{1:s}$ separately with $\epsilon_{\mathbf{A}}$ and $\epsilon_{\mathbf{B}}$. Note that $\epsilon$ defined in (6) is equal to $\max\{\epsilon_{\mathbf{A}}, \epsilon_{\mathbf{B}}\}$. The true Markov transition matrix $\mathbf{T}$ was sampled from a Dirichlet distribution $\mathtt{Dir}((s{-}1){\cdot}\mathbf{I}_s{+}1)$, and we let the approximate $\hat{\mathbf{T}} = \mathbf{T} + \mathbf{E}$, where the perturbation $\mathbf{E} = \eta_{\mathbf{T}}(\mathtt{Dir}((s-1)\cdot\mathbf{I}_s + 1)) - \hat{\mathbf{T}})$ for $\eta_{\mathbf{T}} \in [0,1]$.

We study how the Riccati solution perturbation and sub-optimality gap vary with $\epsilon_{\mathbf{A}}, \epsilon_{\mathbf{B}}, \eta_{\mathbf{T}} \in \{0.01, 0.02, 0.05, 0.1, 0.2, 0.3\}$ and the number of modes $s \in \{10, 20, 30, 40\}$. For each choice of $\epsilon_{\mathbf{A}}$, $\epsilon_{\mathbf{B}}$, and $\eta_{\mathbf{T}}$, we run 100 experiments and record the respective maximums of $\mathbf{\Delta_P} := \max_i \|\hat{\mathbf{P}}_i - \mathbf{P}^\star_i\| / \|\mathbf{P}^\star_i\|$ and $\Delta_J := (\hat{J} - J^\star)/J^\star$ over all 100 runs. In Figures 1 and 2, we have four plots showing $\Delta_{\mathbf{P}}$ and $\Delta_J$ versus uncertainties (i) $\epsilon_{\mathbf{A}}$ ($\epsilon_{\mathbf{B}} = \eta_{\mathbf{T}} = 0$), (ii) $\epsilon_{\mathbf{B}}$ ($\epsilon_{\mathbf{A}} = \eta_{\mathbf{T}} = 0$), (iii) $\eta_{\mathbf{T}}$ ($\epsilon_{\mathbf{A}} = \epsilon_{\mathbf{B}} = 0$), and (iv) $\epsilon = \epsilon_{\mathbf{A}} = \epsilon_{\mathbf{B}} = \eta_{\mathbf{T}}$.

Figure 1 presents four plots that demonstrate how $\Delta_{\mathbf{P}}$ changes as $\epsilon_{\mathbf{A}}$, $\epsilon_{\mathbf{B}}$, $\eta_{\mathbf{T}}$, and $\epsilon$ increase, respectively. Each curve on the plot represents a fixed number of modes $s$. These empirical results are all consistent with (11). In particular, Figure 1 (right) shows that given the uncertainty in the system matrices and in the Markov transition matrix is bounded by $\epsilon$, the perturbation bound to coupled Riccati equations has the rate $\mathcal{O}(\epsilon)$ which degrades linearly as $\epsilon$ increase. Further, it can be easily seen that the gaps indeed increase with the number of modes in the system. Figure 2 demonstrates the relationship between the relative suboptimality $\Delta_J$ and the five parameters $\epsilon_{\mathbf{A}}$, $\epsilon_{\mathbf{B}}$, $\eta_{\mathbf{T}}$, $\epsilon$ and $s$. As can be seen in Figure 2 (right), given the uncertainty in the system matrices and in the Markov transition matrix is bounded by $\epsilon$, the perturbation bounds to the optimal cost decay quadratically which is consistent with our theory.

## V. Conclusions

In this work, we provide a perturbation analysis for cDARE, which arise in the solution of MJS-LQR, and an end-to-end suboptimality guarantee for certainty equivalence control for MJS-LQR. Our results show the robustness of the optimal policy to perturbations in system dynamics and establish the validity of the certainty equivalent control in a neighborhood of the original system. This work opens up multiple future directions. First, with proper system identification algorithms, we can analyze model-based online/adaptive algorithms where control policy is updated continuously over a single trajectory. Second, a natural extension would be to study MJS with output measurements where states are only partially observed, i.e., the LQG setting. This will require considering the dual coupled Riccati equations for filtering.


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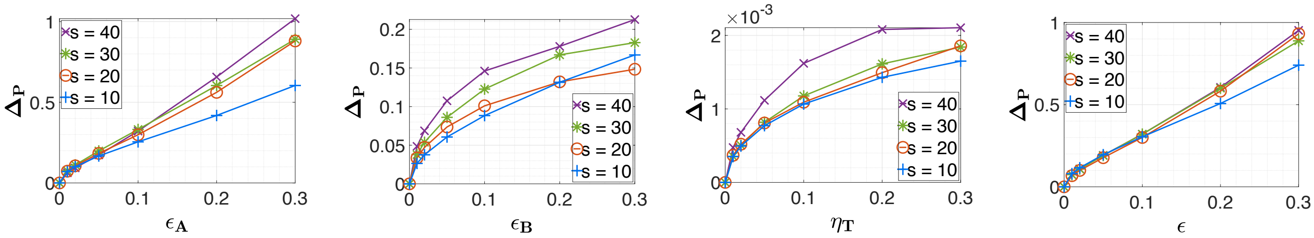


Fig. 1. Riccati solution perturbation. Left to right: $\epsilon_{\mathbf{B}} = \eta_{\mathbf{T}} = 0$, $\epsilon_{\mathbf{A}} = \eta_{\mathbf{T}} = 0$, $\epsilon_{\mathbf{A}} = \epsilon_{\mathbf{B}} = 0$, and $\epsilon = \epsilon_{\mathbf{A}} = \epsilon_{\mathbf{B}} = \eta_{\mathbf{T}}$.

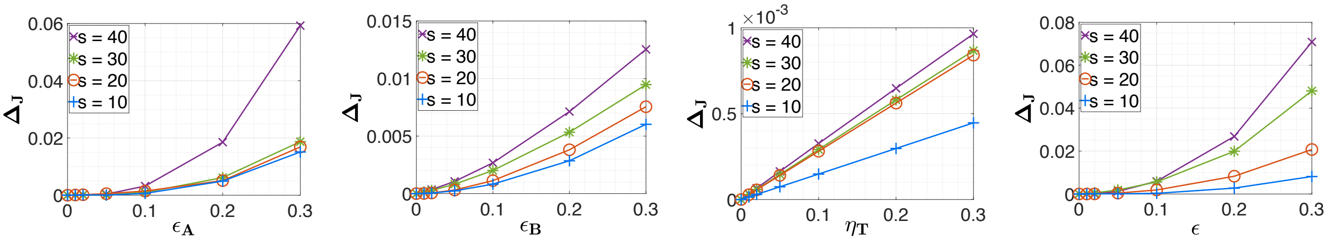


Fig. 2. Sub-optimality gap. Left to right: $\epsilon_{\mathbf{B}} = \eta_{\mathbf{T}} = 0$, $\epsilon_{\mathbf{A}} = \eta_{\mathbf{T}} = 0$, $\epsilon_{\mathbf{A}} = \epsilon_{\mathbf{B}} = 0$, and $\epsilon = \epsilon_{\mathbf{A}} = \epsilon_{\mathbf{B}} = \eta_{\mathbf{T}}$.


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## Appendix

### A. Useful Facts

**Fact 1** (Matrix Facts)**.** *For arbitrary matrices $\mathbf{M}, \mathbf{N}, \mathbf{X}$ with appropriate dimensions, we have the following facts.*

*1) If $\mathbf{M}, \mathbf{N} \succeq 0$, then*

$$\|\mathbf{N}(\mathbf{I} + \mathbf{M}\mathbf{N})^{-1}\| \leq \|\mathbf{N}\|, \tag{14}$$

$$\|(\mathbf{I} + \mathbf{M}\mathbf{N})^{-1}\| \leq 1 + \|\mathbf{N}\|\|\mathbf{M}\|. \tag{15}$$

*2) If $\mathbf{M}$ and $\mathbf{M}+\mathbf{N}$ are invertible, then*

$$\begin{aligned}(\mathbf{M}+\mathbf{N})^{-1} &= \mathbf{M}^{-1} - \mathbf{M}^{-1}\mathbf{N}(\mathbf{M}+\mathbf{N})^{-1} \\ &= \mathbf{M}^{-1} - (\mathbf{M}+\mathbf{N})^{-1}\mathbf{N}\mathbf{M}^{-1}.\end{aligned} \tag{16}$$

*3) If $\mathbf{I}+\mathbf{M}$ and $\mathbf{I}+\mathbf{N}$ are invertible, then*

$$(\mathbf{I}+\mathbf{M})^{-1} - (\mathbf{I}+\mathbf{N})^{-1} = (\mathbf{I}+\mathbf{M})^{-1}(\mathbf{N}-\mathbf{M})(\mathbf{I}+\mathbf{N})^{-1}. \tag{17}$$

*4)* $\mathbf{vec}(\mathbf{M}\mathbf{X}\mathbf{N}) = (\mathbf{N}^\intercal \otimes \mathbf{M})\mathbf{vec}(\mathbf{X}). \quad (18)$

*5) For a collection of matrices $\mathbf{M}_{1:s}$, and for all $i \in [s]$,*

$$\|\varphi_i(\mathbf{M}_{1:s})\| = \|\sum\nolimits_{j=1}^{s} [\mathbf{T}]_{ij}\mathbf{M}_j\| \leq \|\mathbf{M}_{1:s}\|. \tag{19}$$

In Fact 1, (14) is due to [4, Lemma 7] (in their supplement); to see (15), first note that $(\mathbf{I}+\mathbf{M}\mathbf{N})^{-1} = \mathbf{I} - \mathbf{M}\mathbf{N}(\mathbf{I}+\mathbf{M}\mathbf{N})^{-1}$ by matrix inversion lemma, and then apply (14). (16) and (17) also follow from matrix inversion lemma.

**Fact 2.** *For $n{\times}n$ matrices $\mathbf{X}_{1:s}$, let $\mathbf{X} {=} \mathbf{diag}(\mathbf{X}_{1:s})$. Let $\tilde{\mathbf{vec}}(\cdot)$ be the operator that vectorizes all diagonal blocks of $\mathbf{X}$ into a vector, i.e. $\tilde{\mathbf{vec}}(\mathbf{X}) {:=} (\mathbf{vec}(\mathbf{X}_1), \ldots, \mathbf{vec}(\mathbf{X}_s))$. Let $\tilde{\mathbf{vec}}^{-1}$ denote the inverse, i.e. $\tilde{\mathbf{vec}}^{-1}(\tilde{\mathbf{vec}}(\mathbf{X})) = \mathbf{X}$. Then,*

$$\|\tilde{\mathbf{vec}}\| := \sup_{\mathbf{X}=\mathbf{diag}(\mathbf{X}_{1:s}), \|\mathbf{X}\|=1} \|\tilde{\mathbf{vec}}(\mathbf{X})\| \overset{\text{(i)}}{=} \sqrt{ns} \tag{20}$$

$$\|\tilde{\mathbf{vec}}^{-1}\| := \sup_{\|\mathbf{x}\|=1} \|\tilde{\mathbf{vec}}^{-1}(\mathbf{x})\| \overset{\text{(ii)}}{=} 1. \tag{21}$$

Fact 2 follows by noting that (i) achieves the supremum when $\mathbf{X}_i = \mathbf{I}_n$ for all $i$ and (ii) achieves the supremum when $\mathbf{x} = (1, 0, \ldots, 0)$. For a matrix $\mathbf{M}$ and perturbation $\mathbf{\Delta}$, we have the following result adapted from [4, Lemma 5].

**Fact 3.** *Let* $\rho := \rho(\mathbf{M})$ *and* $\tau := \sup_{k\in\mathbb{N}} \|\mathbf{M}^k\|/\rho^k$. *Then,* (a) $\rho(\mathbf{M}+\mathbf{\Delta})\leq\tau\|\mathbf{\Delta}\|+\rho$; (b) $\|(\mathbf{M}+\mathbf{\Delta})^k\| \leq \tau\big(\tau\|\mathbf{\Delta}\|+\rho\big)^k$.

**Fact 4.** *Consider* cDARE$(\mathbf{A}_{1:s},\mathbf{B}_{1:s},\mathbf{T})$ *for a generic* MJS$(\mathbf{A}_{1:s},\mathbf{B}_{1:s},\mathbf{T})$ *and LQR cost matrices* $\mathbf{Q}_{1:s},\mathbf{R}_{1:s}$. *Assume* $\mathbf{Q}_i,\mathbf{R}_i \succ 0$ *for all* $i \in [s]$. *Then, if there exists a positive definite solution* $\mathbf{P}_{1:s}$ *to* cDARE$(\mathbf{A}_{1:s},\mathbf{B}_{1:s},\mathbf{T})$*, then it is the unique solution among* $\{\mathbf{P}_{1:s} : \mathbf{P}_i \succeq 0, \forall\, i \in [s]\}$.

To see this, first note that cDARE$(\mathbf{A}_{1:s},\mathbf{B}_{1:s},\mathbf{T})$ can be written as the Joseph stabilized form [25, (2.2-62)], i.e. $\mathbf{P}_i - \mathbf{L}_i^\intercal\varphi_i(\mathbf{P}_{1:s})\mathbf{L}_i = \mathbf{K}_i^\intercal\mathbf{R}_i\mathbf{K}_i + \mathbf{Q}_i$ where $\mathbf{K}_i = -\big(\mathbf{R}_i + \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}_{1:s})\mathbf{B}_i\big)^{-1}\mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{A}_i$ and $\mathbf{L}_i := \mathbf{A}_i + \mathbf{B}_i\mathbf{K}_i$. Since $\mathbf{Q}_i \succ 0$, we know by Lemma 2 the closed-loop MJS $\mathbf{x}_{t+1}{=}\mathbf{L}_{\omega(t)}\mathbf{x}_t$ is MSS. Then one can obtain Fact 4 by invoking [9, Lemma A.14] which says cDARE has at most one solution with resulting controller stabilizes the MJS.

### B. Proof of Theorem 5

We first provide the road map of the proof.

(a) We construct an operator $\mathcal{K}(\mathbf{X}_{1:s})$ using the difference between the true cDARE$(\mathbf{A}_{1:s},\mathbf{B}_{1:s},\mathbf{T})$ and perturbed cDARE$(\hat{\mathbf{A}}_{1:s},\hat{\mathbf{B}}_{1:s},\hat{\mathbf{T}})$, whose fixed point $\mathbf{X}^\star_{1:s}$ (if exists) guarantees $\hat{\mathbf{P}}_{1:s} := \mathbf{P}^\star_{1:s}+\mathbf{X}^\star_{1:s}$ to be a solution to the perturbed cDARE$(\hat{\mathbf{A}}_{1:s},\hat{\mathbf{B}}_{1:s},\hat{\mathbf{T}})$.

(b) We show when $\epsilon$, $\eta$ are small, $\mathcal{K}(\mathbf{X}_{1:s})$ is a contraction mapping on a closed set $\mathcal{S}_\nu$ whose radius $\nu$ is a function of $\epsilon$ and $\eta$. Thus, there exists a unique fixed point $\mathbf{X}^\star_{1:s} \in \mathcal{S}_\nu$ and $\|\hat{\mathbf{P}}_{1:s} - \mathbf{P}^\star_{1:s}\| = \|\mathbf{X}^\star_{1:s}\| \leq \nu(\epsilon,\eta)$.

(c) Finally, we show $\hat{\mathbf{P}}_{1:s}$ is unique by showing $\hat{\mathbf{P}}_i \succ 0$ and then invoking Fact 4.

*1) Construct operator* $\mathcal{K}$*:* First we define a few notations for the ease of exposition. For all $i \in [s]$, let $\mathbf{S}_i := \mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i^\intercal$ and $\hat{\mathbf{S}}_i := \hat{\mathbf{B}}_i\mathbf{R}_i^{-1}\hat{\mathbf{B}}_i^\intercal$. Define block diagonal matrices $\mathbf{A}$, $\hat{\mathbf{A}}$, $\mathbf{B}$, $\hat{\mathbf{B}}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{P}^\star$, $\hat{\mathbf{P}}$, $\mathbf{K}^\star$, $\mathbf{L}^\star$, $\mathbf{S}$, $\hat{\mathbf{S}}$, $\mathbf{P}$, $\mathbf{X}$, $\mathbf{\Phi}(\mathbf{X})$, $\hat{\mathbf{\Phi}}(\mathbf{X})$ such that their $i$th diagonal blocks are given by $\mathbf{A}_i$, $\hat{\mathbf{A}}_i$, $\mathbf{B}_i$, $\hat{\mathbf{B}}_i$, $\mathbf{Q}_i$, $\mathbf{R}_i$, $\mathbf{P}^\star_i$, $\hat{\mathbf{P}}_i$, $\mathbf{K}^\star_i$, $\mathbf{L}^\star_i$, $\mathbf{S}_i$, $\hat{\mathbf{S}}_i$, $\mathbf{P}_i$, $\mathbf{X}_i$, $\varphi_i(\mathbf{X}_{1:s})$, $\hat{\varphi}_i(\mathbf{X}_{1:s})$ respectively. Note that $\mathbf{P}_i,\mathbf{X}_i \succeq 0$ are just generic variables to be used in function arguments. We will see many equations that hold for each single block also hold for the diagonally concatenated notations.

We have $\mathbf{K}^\star = -(\mathbf{R} + \mathbf{B}^\intercal\mathbf{\Phi}(\mathbf{P}^\star)\mathbf{B})^{-1}\mathbf{B}^\intercal\mathbf{\Phi}(\mathbf{P}^\star)\mathbf{A}$ from (5), then using the matrix inversion lemma, we can get

$$\mathbf{L}^\star = \mathbf{A} + \mathbf{B}\mathbf{K}^\star = (\mathbf{I} + \mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star))^{-1}\mathbf{A}. \tag{22}$$

Furthermore, by diagonally concatenating cDARE (4) and then applying the matrix inversion lemma again, we have

$$\mathbf{X} = \mathbf{A}^\intercal\mathbf{\Phi}(\mathbf{X})(\mathbf{I} + \mathbf{S}\mathbf{\Phi}(\mathbf{X}))^{-1}\mathbf{A} + \mathbf{Q}. \tag{23}$$

Then, we define the following Riccati difference function using the difference between LHS and RHS of (23), with $\mathbf{P}$ as argument and $\mathbf{A},\mathbf{B},\mathbf{T}$ as parameters:

$$\mathcal{F}(\mathbf{P};\mathbf{A},\mathbf{B},\mathbf{T}){:=}\mathbf{P} - \mathbf{A}^\intercal\mathbf{\Phi}(\mathbf{P})(\mathbf{I}{+}\mathbf{S}\mathbf{\Phi}(\mathbf{P}))^{-1}\mathbf{A} - \mathbf{Q}. \tag{24}$$

Though not explicitly listed, $\mathbf{\Phi}$ and $\mathbf{S}$ on the RHS of (24) depend on $\mathbf{T}$ and $\mathbf{B}$ respectively. Since $\mathbf{P}^\star_{1:s}$ is the solution to cDARE$(\mathbf{A}_{1:s},\mathbf{B}_{1:s},\mathbf{T})$, we have $\mathcal{F}(\mathbf{P}^\star;\mathbf{A},\mathbf{B},\mathbf{T}){=}0$. Similarly, if there exists solution $\hat{\mathbf{P}}_{1:s}$ to cDARE$(\hat{\mathbf{A}}_{1:s},\hat{\mathbf{B}}_{1:s},\hat{\mathbf{T}})$, then we also have $\mathcal{F}(\hat{\mathbf{P}};\hat{\mathbf{A}},\hat{\mathbf{B}},\hat{\mathbf{T}}){=}0$.

For $\mathbf{X}$ such that $\mathbf{P}^\star + \mathbf{X} \succeq 0$, we know $\mathbf{I} + \mathbf{S}(\mathbf{P}^\star + \mathbf{X})$ is invertible. Then, for $\mathcal{F}(\mathbf{P}^\star + \mathbf{X};\mathbf{A},\mathbf{B},\mathbf{T})$, we have

$$\begin{aligned}
&\mathcal{F}(\mathbf{P}^\star + \mathbf{X};\mathbf{A},\mathbf{B},\mathbf{T})\\
&\overset{(16)}{=}\mathbf{P}^\star{+}\mathbf{X}{-}\mathbf{A}^\intercal\mathbf{\Phi}(\mathbf{P}^\star{+}\mathbf{X})\cdot\big[(\mathbf{I}{+}\mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star))^{-1}{-}\\
&\quad\underbrace{(\mathbf{I}{+}\mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star{+}\mathbf{X}))^{-1}}_{=:\mathbf{\Gamma}}\mathbf{S}\mathbf{\Phi}(\mathbf{X})(\mathbf{I}{+}\mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star))^{-1}\big]\mathbf{A}{-}\mathbf{Q}\\
&=\mathbf{P}^\star{+}\mathbf{X}{-}\mathbf{A}^\intercal\mathbf{\Phi}(\mathbf{P}^\star{+}\mathbf{X})(\mathbf{I}{-}\mathbf{\Gamma}\mathbf{S}\mathbf{\Phi}(\mathbf{X}))(\mathbf{I}{+}\mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star))^{-1}\mathbf{A}{-}\mathbf{Q}\\
&\overset{(22)}{=}\mathbf{P}^\star{+}\mathbf{X}{-}\mathbf{A}^\intercal\mathbf{\Phi}(\mathbf{P}^\star{+}\mathbf{X})(\mathbf{I}{-}\mathbf{\Gamma}\mathbf{S}\mathbf{\Phi}(\mathbf{X}))\mathbf{L}^\star{-}\mathbf{Q}\\
&\overset{(i)}{=}\mathbf{X}{-}\mathbf{A}^\intercal\mathbf{\Phi}(\mathbf{P}^\star{+}\mathbf{X})(\mathbf{I}{-}\mathbf{\Gamma}\mathbf{S}\mathbf{\Phi}(\mathbf{X}))\mathbf{L}^\star{+}\mathbf{A}^\intercal\mathbf{\Phi}(\mathbf{P}^\star)\mathbf{L}^\star\\
&=\mathbf{X}{-}\mathbf{A}^\intercal\left[\mathbf{\Phi}(\mathbf{P}^\star{+}\mathbf{X})(\mathbf{I}{-}\mathbf{\Gamma}\mathbf{S}\mathbf{\Phi}(\mathbf{X})){-}\mathbf{\Phi}(\mathbf{P}^\star)\right]\mathbf{L}^\star\\
&\overset{(22)}{=}\mathbf{X}{-}\mathbf{L}^{\star\intercal}(\mathbf{I}{+}\mathbf{\Phi}(\mathbf{P}^\star)\mathbf{S})\left[\mathbf{\Phi}(\mathbf{P}^\star{+}\mathbf{X})(\mathbf{I}{-}\mathbf{\Gamma}\mathbf{S}\mathbf{\Phi}(\mathbf{X})){-}\mathbf{\Phi}(\mathbf{P}^\star)\right]\mathbf{L}^\star\\
&=\mathbf{X}{-}\mathbf{L}^{\star\intercal}\underbrace{(\mathbf{I}{+}\mathbf{\Phi}(\mathbf{P}^\star)\mathbf{S})\left[{-}\mathbf{\Phi}(\mathbf{P}^\star)\mathbf{\Gamma}\mathbf{S}{+}\mathbf{I}{-}\mathbf{\Phi}(\mathbf{X})\mathbf{\Gamma}\mathbf{S}\right]}_{=:\mathbf{\Lambda}}\mathbf{\Phi}(\mathbf{X})\mathbf{L}^\star
\end{aligned}$$

where (i) follows from $\mathbf{P}^\star - \mathbf{Q} = \mathbf{A}^\intercal\mathbf{\Phi}(\mathbf{X})\mathbf{L}^\star$ which can be seen from the fact $\mathcal{F}(\mathbf{P}^\star;\mathbf{A},\mathbf{B},\mathbf{T}) = 0$. By expanding $\mathbf{\Lambda}$, one can check $\mathbf{\Lambda} = \mathbf{I} - \mathbf{\Phi}(\mathbf{X})\mathbf{\Gamma}\mathbf{S}$. Plugging this back and using the definition of $\mathbf{\Gamma}$, we have

$$\begin{aligned}
\mathcal{F}(\mathbf{P}^\star + \mathbf{X};\mathbf{A},\mathbf{B},\mathbf{T}) = \mathbf{X} - \mathbf{L}^{\star\intercal}\mathbf{\Phi}(\mathbf{X})\mathbf{L}^\star +\\
\mathbf{L}^{\star\intercal}\mathbf{\Phi}(\mathbf{X})(\mathbf{I} + \mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star) + \mathbf{S}\mathbf{\Phi}(\mathbf{X}))^{-1}\mathbf{S}\mathbf{\Phi}(\mathbf{X})\mathbf{L}^\star.
\end{aligned} \tag{25}$$

If we define

$$\begin{aligned}
&\mathcal{T}(\mathbf{X}){=}\mathbf{X}{-}\mathbf{L}^{\star\intercal}\mathbf{\Phi}(\mathbf{X})\mathbf{L}^\star,\\
&\mathcal{H}(\mathbf{X}){=}\mathbf{L}^{\star\intercal}\mathbf{\Phi}(\mathbf{X})(\mathbf{I}{+}\mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star){+}\mathbf{S}\mathbf{\Phi}(\mathbf{X}))^{-1}\mathbf{S}\mathbf{\Phi}(\mathbf{X})\mathbf{L}^\star,
\end{aligned} \tag{26}$$

we can write $\mathcal{F}(\mathbf{P}^\star + \mathbf{X};\mathbf{A},\mathbf{B},\mathbf{T})$ as

$$\mathcal{F}(\mathbf{P}^\star + \mathbf{X};\mathbf{A},\mathbf{B},\mathbf{T}) = \mathcal{T}(\mathbf{X}) + \mathcal{H}(\mathbf{X}). \tag{27}$$

We now study the invertibility of operator $\mathcal{T}$. Let $\mathbf{Y}_i := \mathbf{X}_i - \mathbf{L}_i^{\star\intercal}\varphi_i(\mathbf{X}_{1:s})\mathbf{L}_i^\star$, and $\mathbf{Y} := \mathbf{diag}(\mathbf{Y}_{1:s})$, then we see $\mathbf{Y} = \mathcal{T}(\mathbf{X}) = \mathbf{X} - \mathbf{L}^{\star\intercal}\mathbf{\Phi}(\mathbf{X})\mathbf{L}^\star$. Apply (18) to $\mathbf{Y}_i$, we have $\mathbf{vec}(\mathbf{Y}_i) = (\mathbf{I}{-}[\mathbf{T}]_{ii}\cdot\mathbf{L}_i^{\star\intercal}{\otimes}\mathbf{L}_i^{\star\intercal})\mathbf{vec}(\mathbf{X}_i){-}\sum_{j\neq i}[\mathbf{T}]_{ij}\mathbf{L}_i^{\star\intercal}\otimes\mathbf{L}_i^{\star\intercal}\mathbf{vec}(\mathbf{X}_j)$. Stacking this equation for all $i$, we have $(\mathbf{I} - \tilde{\mathbf{L}}^\star)\tilde{\mathbf{vec}}(\mathbf{X}) = \tilde{\mathbf{vec}}(\mathbf{Y})$, where $\tilde{\mathbf{vec}}(\cdot)$ is defined in Fact 2. From Sec III, we know $\rho(\tilde{\mathbf{L}}^\star) < 1$, thus $(\mathbf{I}{-}\tilde{\mathbf{L}}^\star)$ is invertible, and inverse operator $\mathcal{T}^{-1}$ exists and is given by

$$\mathbf{X} = \mathcal{T}^{-1}(\mathbf{Y}) = \tilde{\mathbf{vec}}^{-1}\circ(\mathbf{I} - \tilde{\mathbf{L}}^\star)^{-1}\circ\tilde{\mathbf{vec}}(\mathbf{Y}), \tag{28}$$

where $\circ$ denotes operator composition, and $\tilde{\mathbf{vec}}(\cdot)^{-1}$ is defined in Fact 2. With $\mathcal{T}^{-1}$, we define the following operator:

$$\begin{aligned}
\mathcal{K}(\mathbf{X}) := \mathcal{T}^{-1}\big(\mathcal{F}(\mathbf{P}^\star + \mathbf{X};\mathbf{A},\mathbf{B},\mathbf{T}) -\\
\mathcal{F}(\mathbf{P}^\star + \mathbf{X};\hat{\mathbf{A}},\hat{\mathbf{B}},\hat{\mathbf{T}}) - \mathcal{H}(\mathbf{X})\big).
\end{aligned} \tag{29}$$

Suppose there exists a fixed point $\mathbf{X}^\star$ for $\mathcal{K}$, then we see $\mathcal{F}(\mathbf{P}^\star + \mathbf{X}^\star;\hat{\mathbf{A}},\hat{\mathbf{B}},\hat{\mathbf{T}}) = \mathcal{F}(\mathbf{P}^\star + \mathbf{X}^\star;\mathbf{A},\mathbf{B},\mathbf{T}) - \mathcal{T}(\mathbf{X}^\star) - \mathcal{H}(\mathbf{X}^\star) = 0$, i.e. $\hat{\mathbf{P}}_{1:s} = \mathbf{P}^\star_{1:s} + \mathbf{X}^\star_{1:s}$ is a solution to the perturbed cDARE$(\hat{\mathbf{A}}_{1:s},\hat{\mathbf{B}}_{1:s},\hat{\mathbf{T}})$.

*2)* $\mathcal{K}$ *is a Contraction:* We will show $\mathcal{K}(\mathbf{X})$ is a contraction mapping on the closed set

$$\mathcal{S}_\nu := \{\mathbf{X} : \|\mathbf{X}\| \leq \nu, \mathbf{X} = \mathbf{diag}(\mathbf{X}_{1:s}), \mathbf{P}^\star + \mathbf{X} \succeq 0\} \tag{30}$$

so that $\mathcal{K}(\mathbf{X})$ is guaranteed to have a fixed point in $\mathcal{S}_\nu$. To do this, we first present the following lemma (proof in Appendix D) regarding properties of $\mathcal{K}(\mathbf{X})$.

**Lemma 7.** *Assume* $\epsilon \leq \min\{\|\mathbf{B}\|, 1\}$. *Suppose* $\mathbf{X},\mathbf{X}_1,\mathbf{X}_2 \in$

$\mathcal{S}_\nu$ *with* $\nu \le \min\{1, \|\mathbf{S}\|^{-1}\}$, *then*

$$\|\mathcal{K}(\mathbf{X})\| \le \frac{\sqrt{ns}\tau^\star}{1-\rho^\star}\big(\|\mathbf{L}^\star\|^2\|\mathbf{S}\|\nu^2 + \frac{C_\epsilon\epsilon + C_\eta\eta}{2}\big), \quad (31)$$

$$\|\mathcal{K}(\mathbf{X}_1) - \mathcal{K}(\mathbf{X}_2)\| \le \frac{\sqrt{ns}\tau^\star}{1-\rho^\star}\|\mathbf{X}_1 - \mathbf{X}_2\| \cdot \big(3\|\mathbf{L}^\star\|^2\|\mathbf{S}\|\nu + \|\mathbf{B}\|_+^2\|\mathbf{R}^{-1}\|_+ (51\epsilon/C_\epsilon^u + 2\eta/C_\eta^u)\big). \quad (32)$$

To use this lemma, we pick $\nu = \frac{\sqrt{ns}\tau^\star}{1-\rho^\star}(C_\epsilon\epsilon + C_\eta\eta)$. We first show $\mathcal{K}$ maps $\mathcal{S}_\nu$ into itself and then show it is a contraction mapping. Plugging in the upper bounds for $\epsilon$ and $\eta$ in the premises of Theorem 5, we have

$$\nu \le \min\Big\{1, \frac{1}{\|\mathbf{S}\|}, \frac{1-\rho^\star}{12\sqrt{ns}\tau^\star\|\mathbf{L}^\star\|^2\|\mathbf{S}\|}, \frac{\underline{\sigma}(\mathbf{P}^\star)}{12}\Big\}, \quad (33)$$

Following the premise upper bound of $\epsilon$ in Theorem 5 we have $\epsilon \le \min\{\|\mathbf{B}\|, 1\}$. This together with (33) makes Lemma 7 applicable, and we get $\|\mathcal{K}(\mathbf{X})\| \le \frac{1}{12}\nu + \frac{1}{2}\nu = \frac{7}{12}\nu$ by cancelling off $\epsilon$ and $\eta$ in (31) with the definition of $\nu$, and applying the third upper bound for $\nu$ in (33). We know $\nu \le \underline{\sigma}(\mathbf{P}^\star)/12$ from (33), we have $\|\mathcal{K}(\mathbf{X})\| \le \frac{7}{144}\underline{\sigma}(\mathbf{P}^\star)$, thus $\mathbf{P}^\star + \mathcal{K}(\mathbf{X}) \succ 0$. This shows $\mathcal{K}(\mathbf{X}) \in \mathcal{S}_\nu$, i.e. $\mathcal{K}$ maps $\mathcal{S}_\nu$ into itself. Plugging the premise upper bounds for $\epsilon, \eta$ in Theorem 5 and the third upper bound for $\nu$ in (33) into (32) gives $\|\mathcal{K}(\mathbf{X}_1) - \mathcal{K}(\mathbf{X}_2)\| \le \frac{13}{24}\|\mathbf{X}_1 - \mathbf{X}_2\|$, i.e. $\mathcal{K}(\mathbf{X})$ is a contraction mapping on $\mathcal{S}_\nu$, which means $\mathcal{K}(\mathbf{X})$ has a unique fixed point $\mathbf{X}^\star \in \mathcal{S}_\nu$. From the discussion below (29), we know $\hat{\mathbf{P}}_{1:s}$ is a solution to cDARE$(\hat{\mathbf{A}}_{1:s}, \hat{\mathbf{B}}_{1:s}, \hat{\mathbf{T}})$ and $\|\hat{\mathbf{P}}_{1:s} - \mathbf{P}^\star_{1:s}\| = \|\mathbf{X}^\star_{1:s}\| = \|\mathbf{X}^\star\| \le \nu$, which shows (11).

*3) Uniqueness of* $\hat{\mathbf{P}}_{1:s}$*:* Note that $\mathbf{X}^\star \in \mathcal{S}_\nu$ gives $\|\mathbf{X}^\star\| < \nu$, and using (33), we have $\|\mathbf{X}^\star\| < \underline{\sigma}(\mathbf{P}^\star)$, thus $\mathbf{P}^\star + \mathbf{X}^\star \succ 0$. This implies $\hat{\mathbf{P}}_i = \mathbf{P}^\star_i + \mathbf{X}^\star_i \succ 0$ for all $i$. By Fact 4, we know $\hat{\mathbf{P}}_{1:s}$ is the only possible solution to cDARE$(\hat{\mathbf{A}}_{1:s}, \hat{\mathbf{B}}_{1:s}, \hat{\mathbf{T}})$ among $\{\mathbf{X}_{1:s} : \mathbf{X}_i \succeq 0, \forall i\}$. □

### C. Proof of Theorem 6

We first provide the road map of the proof.

(a) We bound the controller difference $\|\mathbf{K}^\star_{1:s} - \hat{\mathbf{K}}_{1:s}\|$ in terms of $\|\hat{\mathbf{P}}_{1:s} - \mathbf{P}^\star_{1:s}\|$ and provide conditions under which $\hat{\mathbf{K}}_{1:s}$ stabilizes the true MJS$(\mathbf{A}_{1:s}, \mathbf{B}_{1:s}, \mathbf{T})$.

(b) For $\hat{J}$ incurred by the stabilizing $\hat{\mathbf{K}}_{1:s}$, we bound the suboptimality gap $\hat{J} - J^\star$ in terms of $\|\mathbf{K}^\star_{1:s} - \hat{\mathbf{K}}_{1:s}\|$.

(c) We Combine steps (a), (b) and Theorem 5 to obtain the final result.

*1) Properties of* $\hat{\mathbf{K}}_{1:s}$*:* We show that when $\hat{\mathbf{P}}_{1:s}$ is close to $\mathbf{P}_{1:s}$, then $\hat{\mathbf{K}}_{1:s}$ is also close to $\mathbf{K}_{1:s}$.

**Lemma 8** (Controller mismatch)**.** *Suppose* $\|\hat{\mathbf{P}}_{1:s} - \mathbf{P}^\star_{1:s}\| \le f(\epsilon,\eta)$ *for some function* $f(\epsilon,\eta)$ *such that* $\max\{\epsilon,\eta\} \le f(\epsilon,\eta) \le \Gamma_\star$*. Then, under Assumption 3, we have*

$$\|\mathbf{K}^\star_{1:s} - \hat{\mathbf{K}}_{1:s}\| \le 28\Gamma_\star^3 \frac{(\underline{\sigma}(\mathbf{R}_{1:s}) + \Gamma_\star^3)}{\underline{\sigma}(\mathbf{R}_{1:s})^2} f(\epsilon,\eta) \quad (34)$$

*Proof.* Recall $\mathbf{K}^\star_i = -\big(\mathbf{R}_i + \mathbf{B}_i^\intercal \varphi_i(\mathbf{P}^\star_{1:s})\mathbf{B}_i\big)^{-1}\mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{A}_i$ and $\hat{\mathbf{K}}_i = -\big(\mathbf{R}_i + \hat{\mathbf{B}}_i^\intercal\hat{\varphi}_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{B}}_i\big)^{-1}\hat{\mathbf{B}}_i^\intercal\hat{\varphi}_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{A}}_i$. As an auxiliary step, we define $\tilde{\mathbf{K}}_i := -\big(\mathbf{R}_i + \hat{\mathbf{B}}_i^\intercal\varphi_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{B}}_i\big)^{-1} \cdot \hat{\mathbf{B}}_i^\intercal\varphi_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{A}}_i$. Then, we have

$$\|\mathbf{K}^\star_i - \hat{\mathbf{K}}_i\| \le \|\mathbf{K}^\star_i - \tilde{\mathbf{K}}_i\| + \|\tilde{\mathbf{K}}_i - \hat{\mathbf{K}}_i\|. \quad (35)$$

Note that $\|\mathbf{K}^\star_{1:s} - \hat{\mathbf{K}}_{1:s}\| = \max_i \|\mathbf{K}^\star_i - \hat{\mathbf{K}}_i\|$, thus it suffices to bound $\|\mathbf{K}^\star_i - \tilde{\mathbf{K}}_i\|$ and $\|\tilde{\mathbf{K}}_i - \hat{\mathbf{K}}_i\|$ respectively. For $\|\mathbf{K}^\star_i - \tilde{\mathbf{K}}_i\|$, we can see $\|\mathbf{K}^\star_i - \tilde{\mathbf{K}}_i\| \le M\delta_N + \delta_M N$ where

$$M = \|\big(\mathbf{R}_i + \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{B}_i\big)^{-1}\|, \quad N = \|\hat{\mathbf{B}}_i^\intercal\varphi_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{A}}_i\|$$
$$\delta_M = \|\big(\mathbf{R}_i + \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{B}_i\big)^{-1} - \big(\mathbf{R}_i + \hat{\mathbf{B}}_i^\intercal\varphi_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{B}}_i\big)^{-1}\|$$
$$\delta_N = \|\mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{A}_i - \hat{\mathbf{B}}_i^\intercal\varphi_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{A}}_i\|$$

We next upper bound $M, \delta_N, \delta_M$, and $N$. Since we assume $\mathbb{R}_i \succ 0$, it is easy to see $M = \|\big(\mathbf{R}_i + \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{B}_i\big)^{-1}\| \le \frac{1}{\underline{\sigma}(\mathbf{R}_i)}$. For $\delta_N$, let $\Delta_{\mathbf{A}_i} = \hat{\mathbf{A}}_i - \mathbf{A}_i$, $\Delta_{\mathbf{B}_i} = \hat{\mathbf{B}}_i - \mathbf{B}_i$, $\Delta_{\mathbf{P}_i} = \hat{\mathbf{P}}_i - \mathbf{P}^\star_i$, then we have

$$\begin{aligned}
\delta_N &= \|\mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{A}_i - \hat{\mathbf{B}}_i^\intercal\varphi_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{A}}_i\| \\
&= \|\mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{A}_i - (\mathbf{B}_i + \Delta_{\mathbf{B}_i})^\intercal \\
&\quad \cdot \big[\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{A}_i + \varphi_i(\Delta_{\mathbf{P}^\star_{1:s}})\mathbf{A}_i + \varphi_i(\mathbf{P}^\star_{1:s})\Delta_{\mathbf{A}_i} + \varphi_i(\Delta_{\mathbf{P}^\star_{1:s}})\Delta_{\mathbf{A}_i}\big]\| \\
&= \|\mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{A}_i - \big[\mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{A}_i + \mathbf{B}_i^\intercal\varphi_i(\Delta_{\mathbf{P}^\star_{1:s}})\mathbf{A}_i \\
&\quad + \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\Delta_{\mathbf{A}_i} + \mathbf{B}_i^\intercal\varphi_i(\Delta_{\mathbf{P}^\star_{1:s}})\Delta_{\mathbf{A}_i} + \Delta_{\mathbf{B}_i}^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{A}_i \\
&\quad + \Delta_{\mathbf{B}_i}^\intercal\varphi_i(\Delta_{\mathbf{P}^\star_{1:s}})\mathbf{A}_i + \Delta_{\mathbf{B}_i}^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\Delta_{\mathbf{A}_i} + \Delta_{\mathbf{B}_i}^\intercal\varphi_i(\Delta_{\mathbf{P}^\star_{1:s}})\Delta_{\mathbf{A}_i}\big]\| \\
&\overset{(19)}{\le} \|\mathbf{A}_i\|\|\mathbf{B}_i\|f(\epsilon,\eta) + \|\mathbf{B}_i\|\|\mathbf{P}^\star_{1:s}\|\epsilon + \|\mathbf{B}_i\|f(\epsilon,\eta)\epsilon \\
&\quad + \|\mathbf{A}_i\|\|\mathbf{P}^\star_{1:s}\|\epsilon + \|\mathbf{A}_i\|f(\epsilon,\eta)\epsilon + \|\mathbf{P}^\star_{1:s}\|\epsilon^2 + f(\epsilon,\eta)\epsilon^2, \\
&\le 3\Gamma_\star^2 f(\epsilon,\eta),
\end{aligned}$$

where the last line follows from the assumption that $\epsilon < f(\epsilon,\eta)$. For $\delta_M$, we have

$$\begin{aligned}
\delta_M &= \|\big(\mathbf{R}_i + \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{B}_i\big)^{-1} - \big(\mathbf{R}_i + \hat{\mathbf{B}}_i^\intercal\varphi_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{B}}_i\big)^{-1}\| \\
&\overset{(16)}{\le} \|\big(\mathbf{R}_i + \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{B}_i\big)^{-1}\| \cdot \|\big(\mathbf{R}_i + \hat{\mathbf{B}}_i^\intercal\varphi_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{B}}_i\big)^{-1}\| \\
&\quad \cdot \|\hat{\mathbf{B}}_i^\intercal\varphi_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{B}}_i - \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{B}_i\| \\
&\le \frac{3\Gamma_\star^2 f(\epsilon,\eta)}{\underline{\sigma}(\mathbf{R}_i)^2}.
\end{aligned}$$

Similarly, we have the following for $N$.

$$\begin{aligned}
N &= \|\hat{\mathbf{B}}_i^\intercal\varphi_i(\hat{\mathbf{P}}_{1:s})\hat{\mathbf{A}}_i\| \\
&= \|\mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{A}_i + \mathbf{B}_i^\intercal\varphi_i(\Delta_{\mathbf{P}^\star_{1:s}})\mathbf{A}_i + \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\Delta_{\mathbf{A}_i} \\
&\quad + \mathbf{B}_i^\intercal\varphi_i(\Delta_{\mathbf{P}^\star_{1:s}})\Delta_{\mathbf{A}_i} + \Delta_{\mathbf{B}_i}^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{A}_i + \Delta_{\mathbf{B}_i}^\intercal\varphi_i(\Delta_{\mathbf{P}^\star_{1:s}})\mathbf{A}_i \\
&\quad + \Delta_{\mathbf{B}_i}^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\Delta_{\mathbf{A}_i} + \Delta_{\mathbf{B}_i}^\intercal\varphi_i(\Delta_{\mathbf{P}^\star_{1:s}})\Delta_{\mathbf{A}_i}\| \\
&\le \big(\|\mathbf{A}_i\|\|\mathbf{B}_i\| + \|\mathbf{A}_i\|\|\mathbf{P}^\star_{1:s}\| + \|\mathbf{B}_i\|\|\mathbf{P}^\star_{1:s}\| + \|\mathbf{A}_i\|\epsilon + \|\mathbf{B}_i\|\epsilon \\
&\quad + \|\mathbf{P}^\star_{1:s}\|\epsilon + \epsilon^2\big) \cdot f(\epsilon,\eta) + \|\mathbf{A}_i\|\|\mathbf{B}_i\|\|\mathbf{P}^\star_{1:s}\| \\
&\le 3\Gamma_\star^2 f(\epsilon,\eta) + \Gamma_\star^3.
\end{aligned}$$

Combining the bounds for $M, \delta_N, \delta_M$, and $N$ we obtained thus far, we have $\|\mathbf{K}^\star_i - \tilde{\mathbf{K}}_i\| \le 12\Gamma_\star^2\frac{(\underline{\sigma}(\mathbf{R}_i) + \Gamma_\star^3)}{\underline{\sigma}(\mathbf{R}_i)^2}f(\epsilon,\eta)$. Using similar techniques, for the other term on the RHS of (35), we can show $\|\tilde{\mathbf{K}}_i - \hat{\mathbf{K}}_i\| \le 16\Gamma_\star^3\frac{(\underline{\sigma}(\mathbf{R}_i) + \Gamma_\star^3)}{\underline{\sigma}(\mathbf{R}_i)^2}\eta$. Recall we assume $\eta \le f(\epsilon,\eta)$, then by triangle inequality, we have $\|\mathbf{K}^\star_i - \hat{\mathbf{K}}_i\| \le 28\Gamma_\star^3\frac{(\underline{\sigma}(\mathbf{R}_i) + \Gamma_\star^3)}{\underline{\sigma}(\mathbf{R}_i)^2}f(\epsilon,\eta)$. □

For $\hat{\mathbf{K}}_{1:s}$, let $\hat{\mathbf{L}}_i := \mathbf{A}_i + \mathbf{B}_i\hat{\mathbf{K}}_i$ and define the augmented closed-loop state matrix $\tilde{\mathbf{L}}^\circ \in \mathbb{R}^{sn^2 \times sn^2}$ with $ij$-th $n^2 \times n^2$ block given by $[\tilde{\mathbf{L}}^\circ]_{ij} := [\mathbf{T}]_{ij}\hat{\mathbf{L}}_i^\intercal \otimes \hat{\mathbf{L}}_i^\intercal$.

**Lemma 9** (Stabilizability of $\hat{\mathbf{K}}$)**.** *Suppose* $\|\hat{\mathbf{K}}_{1:s} - \mathbf{K}^\star_{1:s}\| \le \frac{1-\rho^\star}{2\sqrt{s}\tau^\star(1+2\|\mathbf{L}^\star_{1:s}\|)\|\mathbf{B}_{1:s}\|} =: \bar{\epsilon}_{\mathbf{K}}$*, then (a)* $\rho(\tilde{\mathbf{L}}^\circ) < \frac{1+\rho^\star}{2}$*, i.e.* $\hat{\mathbf{K}}_{1:s}$ *is a stabilizing controller; (b)* $\|(\tilde{\mathbf{L}}^\circ)^k\| \le \tau^\star(\frac{1+\rho^\star}{2})^k$*;*

*Proof.* What remains to show is that $\hat{\mathbf{K}}_{1:s}$ stabilizes the true MJS. We let $\mathbf{\Delta}_{\mathbf{K}_i} := \hat{\mathbf{K}}_i - \mathbf{K}^\star_i$ and $\hat{\mathbf{L}}_i := \mathbf{A}_i + \mathbf{B}_i\hat{\mathbf{K}}_i$, then we see $\hat{\mathbf{L}}_i = \mathbf{L}^\star_i + \mathbf{B}_i\mathbf{\Delta}_{\mathbf{K}_i}$. Under controller $\hat{\mathbf{K}}_{1:s}$, we define the augmented closed-loop state matrix $\tilde{\mathbf{L}}^\circ \in \mathbb{R}^{sn^2 \times sn^2}$ with $ij$-th $n^2 \times n^2$ block given by $[\tilde{\mathbf{L}}^\circ]_{ij} := [\mathbf{T}]_{ij}\hat{\mathbf{L}}_i^\intercal \otimes \hat{\mathbf{L}}_i^\intercal$. Note that

$[\tilde{\mathbf{L}}^\circ]_{ij} - [\tilde{\mathbf{L}}^\star]_{ij} = [\mathbf{T}]_{ij}((\mathbf{B}_i\mathbf{\Delta}_{\mathbf{K}_i})^\intercal \otimes (\mathbf{B}_i\mathbf{\Delta}_{\mathbf{K}_i})^\intercal + (\mathbf{B}_i\mathbf{\Delta}_{\mathbf{K}_i})^\intercal \otimes \mathbf{L}_i^{\star\intercal} + \mathbf{L}_i^{\star\intercal} \otimes (\mathbf{B}_i\mathbf{\Delta}_{\mathbf{K}_i})^\intercal)$, then $\|[\tilde{\mathbf{L}}^\circ]_{ij} - [\tilde{\mathbf{L}}^\star]_{ij}\| \leq [\mathbf{T}]_{ij}(\|\mathbf{B}_i\|^2 \cdot \|\mathbf{\Delta}_{\mathbf{K}_i}\|^2 + 2\|\mathbf{B}_i\|\|\mathbf{L}_i^\star\|\|\mathbf{\Delta}_{\mathbf{K}_i}\|) \leq [\mathbf{T}]_{ij}(1+2\|\mathbf{L}_i\|)\|\mathbf{B}_i\|\|\mathbf{\Delta}_{\mathbf{K}_i}\| \leq [\mathbf{T}]_{ij}\frac{1-\rho^\star}{2\sqrt{s}\tau^\star}$. Using Cauchy-Schwartz inequality, we have $\|\tilde{\mathbf{L}}^\circ - \tilde{\mathbf{L}}^\star\| \leq \big(\sum_{i,j}\|[\tilde{\mathbf{L}}^\circ]_{ij} - [\tilde{\mathbf{L}}^\star]_{ij}\|^2\big)^{0.5} \leq \frac{1-\rho^\star}{2\tau^\star}$. Finally, we can conclude the proof by invoking Fact 3. □

*2) $\hat{J} - J^\star$ vs. $\|\mathbf{K}^\star_{1:s} - \hat{\mathbf{K}}_{1:s}\|$:* Adapting [21, Lemma 3-(2)] to noisy MJS and infinite-horizon average cost case, we have the following result.

**Lemma 10.** *Suppose $\mathbf{K}_{1:s}$ is a stabilizing controller. Let $\mathbf{\Pi}_i := \pi_\infty(i)\mathbf{I}_n$ and $\mathbf{\Pi} := \mathbf{diag}(\mathbf{\Pi}_{1:s})$. Let $\mathbf{\Sigma}_{1:s}$ be the solution to $\tilde{\mathbf{vec}}(\mathbf{\Sigma}) = \tilde{\mathbf{L}}^{\circ\intercal}\tilde{\mathbf{vec}}(\mathbf{\Sigma}) + \sigma_\mathbf{w}^2\tilde{\mathbf{vec}}(\mathbf{\Pi})$, where $\mathbf{\Sigma} := \mathbf{diag}(\mathbf{\Sigma}_{1:s})$. Then, $\hat{J} - J^\star = \sum_i \mathbf{tr}\big(\mathbf{\Sigma}_i(\hat{\mathbf{K}}_i - \mathbf{K}_i^\star)^\intercal(\mathbf{R}_i + \mathbf{B}_i^\intercal\varphi_i(\mathbf{P}^\star_{1:s})\mathbf{B}_i(\hat{\mathbf{K}}_i - \mathbf{K}_i^\star))\big)$*

In Lemma 10, the equation described by $\mathbf{\Sigma}$ is essentially the coupled Lyapunov equation for MJS, and it can be shown $\mathbf{\Sigma}_i = \lim_{t\to\infty}\mathbb{E}[\mathbf{x}_t\mathbf{x}_t^\intercal \mathbf{1}_{\omega(t)=i}]$ where $\mathbf{x}_t$ is the state under controller $\hat{\mathbf{K}}_{1:s}$. Combining Lemma 9 and 10, we have

**Corollary 11.** *Suppose $\|\hat{\mathbf{K}}_{1:s} - \mathbf{K}^\star_{1:s}\| \leq \bar{\epsilon}_\mathbf{K}$, then*

$$\hat{J} - J^\star \leq \frac{2\sigma_\mathbf{w}^2 s^{1.5}\sqrt{n}\min\{n,p\}\tau^\star}{1-\rho^\star}(\|\mathbf{R}_{1:s}\| + \Gamma_\star^3)\|\hat{\mathbf{K}}_{1:s} - \mathbf{K}^\star_{1:s}\|^2.$$

*Proof.* We first bound $\|\mathbf{\Sigma}_i\|$ in Lemma 10. Similar to (28), we have $\mathbf{\Sigma} = \sigma_\mathbf{w}^2 \cdot \tilde{\mathbf{vec}}^{-1} \circ (\mathbf{I} - \tilde{\mathbf{L}}^{\circ\intercal})^{-1} \circ \tilde{\mathbf{vec}}(\mathbf{\Pi})$. Using Fact (2) and the sub-multiplicative property of operator norms, we have $\|\mathbf{\Sigma}_i\| \leq \|\mathbf{\Sigma}\| \leq \sqrt{ns}\|(\mathbf{I} - \tilde{\mathbf{L}}^{\circ\intercal})^{-1}\|\|\mathbf{\Pi}\|$. Note that $\|\mathbf{\Pi}\| \leq 1$ and $\|(\mathbf{I} - \tilde{\mathbf{L}}^{\circ\intercal})^{-1}\| = \|\sum_{k=0}^\infty (\tilde{\mathbf{L}}^\circ)^k\| \leq \sum_{k=0}^\infty \|(\tilde{\mathbf{L}}^\circ)^k\| \leq \frac{2\tau^\star}{1-\rho^\star}$, where the last inequality follows from Lemma 9 (b). Thus, $\|\mathbf{\Sigma}_i\| \leq \frac{2\sigma_\mathbf{w}^2\sqrt{sn}\tau^\star}{1-\rho^\star}$. Then, Lemma 10 gives $\hat{J} - J^\star \leq s\|\mathbf{\Sigma}_i\|(\|\mathbf{R}_{1:s}\| + \|\mathbf{B}_{1:s}\|^2\|\mathbf{P}^\star\|)\|\hat{\mathbf{K}}_{1:s} - \mathbf{K}^\star_{1:s}\|_F^2 \leq \frac{2\sigma_\mathbf{w}^2 s^{1.5}\sqrt{n}\min\{n,p\}\tau^\star}{1-\rho^\star}(\|\mathbf{R}_{1:s}\| + \Gamma_\star^3)\|\hat{\mathbf{K}}_{1:s} - \mathbf{K}^\star_{1:s}\|^2$. □

*3) Proof of Theorem 6:* To prove Theorem 6, we only need to combine Theorem 5, Lemma 8, and Corollary 11. By Theorem 5, we can choose $f(\epsilon,\eta) := \frac{\sqrt{ns}\tau^\star}{1-\rho^\star}(C_\epsilon\epsilon + C_\eta\eta)$ in Lemma 8. The, when $C_\epsilon\epsilon + C_\eta\eta \leq \frac{(1-\rho^\star)\min\{\Gamma_\star, \underline{\sigma}(\mathbf{R}_{1:s})^2\bar{\epsilon}_\mathbf{K}\}}{28\sqrt{ns}\tau^\star\Gamma_\star^3(\underline{\sigma}(\mathbf{R}_{1:s})+\Gamma_\star^3)}$, the premise conditions $\max\{\epsilon,\eta\} \leq f(\epsilon,\eta) \leq \Gamma_\star$ in Lemma 8 and $\|\hat{\mathbf{K}}_{1:s} - \mathbf{K}^\star_{1:s}\| \leq \bar{\epsilon}_\mathbf{K}$ in Corollary 11 hold. Theorem 5 and Lemma 8 give $\|\mathbf{K}^\star_{1:s} - \hat{\mathbf{K}}_{1:s}\| \leq 28\sqrt{ns}\tau^\star\Gamma_\star^3\frac{(\underline{\sigma}(\mathbf{R}_{1:s})+\Gamma_\star^3)}{(1-\rho^\star)\underline{\sigma}(\mathbf{R}_{1:s})^2}(C_\epsilon\epsilon + C_\eta\eta)$ which shows (12). Combining this with Corollary 11 shows (13). □

### D. Proof of Lemma 7

To ease the exposition, let $\mathbf{P}^\star_\mathbf{X} := \mathbf{P}^\star + \mathbf{X}$ and define $\mathbf{P}^\star_{\mathbf{X}_1}, \mathbf{P}^\star_{\mathbf{X}_1}$ similarly. Let $\mathbf{\Delta}_\mathbf{A} := \hat{\mathbf{A}} - \mathbf{A}$, $\mathbf{\Delta}_\mathbf{B} := \hat{\mathbf{B}} - \mathbf{B}$, and $\mathbf{\Delta}_\mathbf{S} := \hat{\mathbf{S}} - \mathbf{S}$. We list a few preliminary results (when $\mathbf{X} \in \mathcal{S}_\nu$) to be used later.

- $\|\mathbf{\Phi}(\mathbf{X})\| \leq \|\mathbf{X}\|, \quad \|\hat{\mathbf{\Phi}}(\mathbf{X})\| \leq \|\mathbf{X}\|.$ (36)
- $\|\mathbf{\Phi}(\mathbf{X}) - \hat{\mathbf{\Phi}}(\mathbf{X})\| \leq \eta\|\mathbf{X}\|.$ (37)
- $\max\{\|\mathbf{P}^\star_\mathbf{X}\|, \|\mathbf{\Phi}(\mathbf{P}^\star_\mathbf{X})\|, \|\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X})\|\} \leq \|\mathbf{P}^\star\|_+.$ (38)
- $\|\mathbf{S}\| \leq \|\mathbf{B}\|^2\|\mathbf{R}^{-1}\|, \|\mathbf{\Delta}_\mathbf{S}\| \leq 3\|\mathbf{B}\|\|\mathbf{R}^{-1}\|\epsilon,$ $\|\hat{\mathbf{S}}\| \leq 4\|\mathbf{B}\|^2\|\mathbf{R}^{-1}\|$ (39)
- $\max\left\{\|(\mathbf{I} + \mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star_\mathbf{X}))^{-1}\|, \|(\mathbf{I} + \mathbf{S}\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X}))^{-1}\|\right\}$ $\leq \|\mathbf{B}\|_+^2\|\mathbf{R}^{-1}\|_+\|\mathbf{P}^\star\|_+$ (40)
- $\max\left\{\|(\mathbf{I} + \hat{\mathbf{S}}\mathbf{\Phi}(\mathbf{P}^\star_\mathbf{X}))^{-1}\|, \|(\mathbf{I} + \hat{\mathbf{S}}\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X}))^{-1}\|\right\}$ $\leq 4\|\mathbf{B}\|_+^2\|\mathbf{R}^{-1}\|_+\|\mathbf{P}^\star\|_+$. (41)

(38) is due to $\nu \leq 1$, and (39) uses $\|\mathbf{\Delta}_\mathbf{B}\| \leq \epsilon \leq \|\mathbf{B}\|$. (40) and (41) follows from (15), (38), (39).

Now, we are ready to begin the main proof. We first define

$$\mathcal{G}_1(\mathbf{X}) := \mathcal{F}(\mathbf{P}^\star_\mathbf{X}; \mathbf{A}, \mathbf{B}, \hat{\mathbf{T}}) - \mathcal{F}(\mathbf{P}^\star_\mathbf{X}; \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{T}})$$
$$\mathcal{G}_2(\mathbf{X}) := \mathcal{F}(\mathbf{P}^\star_\mathbf{X}; \mathbf{A}, \mathbf{B}, \mathbf{T}) - \mathcal{F}(\mathbf{P}^\star_\mathbf{X}; \mathbf{A}, \mathbf{B}, \hat{\mathbf{T}}).$$

Then, we have the following decomposition.

$$\mathcal{K}(\mathbf{X}) = \mathcal{T}^{-1}(\mathcal{G}_1(\mathbf{X}) + \mathcal{G}_2(\mathbf{X}) - \mathcal{H}(\mathbf{X})), \quad (42)$$
$$\mathcal{K}(\mathbf{X}_1) - \mathcal{K}(\mathbf{X}_2) = \mathcal{T}^{-1}(\mathcal{G}_1(\mathbf{X}_1) - \mathcal{G}_1(\mathbf{X}_2) + \mathcal{G}_2(\mathbf{X}_1) - \mathcal{G}_2(\mathbf{X}_2) - \mathcal{H}(\mathbf{X}_1) + \mathcal{H}(\mathbf{X}_2)) \quad (43)$$

To bound the $\|\mathcal{K}(\mathbf{X})\|$ and $\|\mathcal{K}(\mathbf{X}_1) - \mathcal{K}(\mathbf{X}_2)\|$, we will bound $\|\mathcal{T}^{-1}\|$, $\|\mathcal{H}(\mathbf{X})\|$, $\|\mathcal{G}_1(\mathbf{X})\|$, $\|\mathcal{G}_2(\mathbf{X})\|$, $\|\mathcal{H}(\mathbf{X}_1) - \mathcal{H}(\mathbf{X}_2)\|$, $\|\mathcal{G}_1(\mathbf{X}_1) - \mathcal{G}_1(\mathbf{X}_2)\|$, $\|\mathcal{G}_2(\mathbf{X}_1) - \mathcal{G}_2(\mathbf{X}_2)\|$ individually, for any $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2 \in \mathcal{S}_\nu$ and then combine them using triangle inequality and operator composition sub-multiplicativity, i.e.

$$\|\mathcal{K}(\mathbf{X})\| \leq \|\mathcal{T}^{-1}\|(\|\mathcal{G}_1(\mathbf{X})\| + \|\mathcal{G}_2(\mathbf{X})\| + \|\mathcal{H}(\mathbf{X})\|) \quad (44)$$
$$\|\mathcal{K}(\mathbf{X}_1) - \mathcal{K}(\mathbf{X}_2)\| \leq \|\mathcal{T}^{-1}\|(\|\mathcal{G}_1(\mathbf{X}_1) - \mathcal{G}_1(\mathbf{X}_2)\| + \|\mathcal{G}_2(\mathbf{X}_1) - \mathcal{G}_2(\mathbf{X}_2)\| + \|\mathcal{H}(\mathbf{X}_1) - \mathcal{H}(\mathbf{X}_2)\|) \quad (45)$$

*1) Bound $\|\mathcal{K}(\mathbf{X})\|$:* By the definition of $\mathcal{T}^{-1}$ in (28), we know $\mathcal{T}^{-1}(\mathbf{Y}) = \tilde{\mathbf{vec}}^{-1} \circ (\mathbf{I} - \tilde{\mathbf{L}}^\star)^{-1} \circ \tilde{\mathbf{vec}}(\mathbf{Y})$. Then, for $\|\mathcal{T}^{-1}\|$, similar to the proof for Corollary 11, we have $\|\mathcal{T}^{-1}\| \leq \frac{\sqrt{sn}\tau^\star}{1-\rho^\star}$. By definition of $\mathcal{H}(\mathbf{X})$ in (27), we have $\|\mathcal{H}(\mathbf{X})\| \leq \|\mathbf{L}^\star\|^2\|\mathbf{S}\|\|\mathbf{X}\|^2 \leq \|\mathbf{L}^\star\|^2\|\mathbf{S}\|\nu^2$, where (14) and (36) are used. For term $\mathcal{G}_1(\mathbf{X})$, using (17), we can decompose it as

$$\begin{aligned}\mathcal{G}_1(\mathbf{X}) = \\ &-\mathbf{A}^\intercal\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X})(\mathbf{I} + \mathbf{S}\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X}))^{-1}\mathbf{\Delta}_\mathbf{S}\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X})(\mathbf{I} + \hat{\mathbf{S}}\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X}))^{-1}\mathbf{A} \\ &+\mathbf{\Delta}_\mathbf{A}^\intercal\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X})(\mathbf{I} + \hat{\mathbf{S}}\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X}))^{-1}\mathbf{A} + \mathbf{A}^\intercal\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X})(\mathbf{I} + \hat{\mathbf{S}}\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X}))^{-1}\mathbf{\Delta}_\mathbf{A} \\ &+\mathbf{\Delta}_\mathbf{A}^\intercal\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X})(\mathbf{I} + \hat{\mathbf{S}}\hat{\mathbf{\Phi}}(\mathbf{P}^\star_\mathbf{X}))^{-1}\mathbf{\Delta}_\mathbf{A}.\end{aligned}$$

With properties (14), (38), (39), and the premise assumption $\epsilon \leq \|\mathbf{B}\|$, we can show $\|\mathcal{G}_1(\mathbf{X})\| \leq 3\|\mathbf{A}\|_+^2\|\mathbf{B}\|_+ \cdot \|\mathbf{P}^\star\|_+^2\|\mathbf{R}^{-1}\|_+\epsilon$. Similarly, we can show $\|\mathcal{G}_2(\mathbf{X})\| \leq \|\mathbf{A}\|_+^2 \cdot \|\mathbf{B}\|_+^4\|\mathbf{P}^\star\|_+^3\|\mathbf{R}^{-1}\|_+^2\eta$ by invoking (14), (17), (37), (38), (39), and (40). Finally, using the relation in (44), we can show the upper bound for $\|\mathcal{K}(\mathbf{X})\|$ in (31).

*2) Bound $\|\mathcal{K}(\mathbf{X}_1) - \mathcal{K}(\mathbf{X}_2)\|$:* We first derive bounds for $\|\mathcal{H}(\mathbf{X}_1) - \mathcal{H}(\mathbf{X}_2)\|$, $\|\mathcal{G}_1(\mathbf{X}_1) - \mathcal{G}_1(\mathbf{X}_2)\|$, and $\|\mathcal{G}_2(\mathbf{X}_1) - \mathcal{G}_2(\mathbf{X}_2)\|$. With the help of (17), the following can be obtained.

$$\begin{aligned}\mathcal{H}(\mathbf{X}_1) - \mathcal{H}(\mathbf{X}_2) = \mathbf{L}^{\star\intercal}\mathbf{\Phi}(\mathbf{X}_1)(\mathbf{I} + \mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star_{\mathbf{X}_1}))^{-1} \\ \cdot\mathbf{S}\mathbf{\Phi}(\mathbf{X}_2 - \mathbf{X}_1)(\mathbf{I} + \mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star_{\mathbf{X}_2}))^{-1}\mathbf{S}\mathbf{\Phi}(\mathbf{X}_1)\mathbf{L}^\star \\ -\mathbf{L}^{\star\intercal}\mathbf{\Phi}(\mathbf{X}_2 - \mathbf{X}_1)(\mathbf{I} + \mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star_{\mathbf{X}_2}))^{-1}\mathbf{S}\mathbf{\Phi}(\mathbf{X}_2)\mathbf{L}^\star \\ -\mathbf{L}^{\star\intercal}\mathbf{\Phi}(\mathbf{X}_1)(\mathbf{I} + \mathbf{S}\mathbf{\Phi}(\mathbf{P}^\star_{\mathbf{X}_2}))^{-1}\mathbf{S}\mathbf{\Phi}(\mathbf{X}_2 - \mathbf{X}_1)\mathbf{L}^\star.\end{aligned} \quad (46)$$

Using (14), (36), and $\nu \leq \|\mathbf{S}\|^{-1}$, we have $\|\mathcal{H}(\mathbf{X}_1) - \mathcal{H}(\mathbf{X}_2)\| \leq 3\|\mathbf{L}^\star\|^2\|\mathbf{S}\|\nu\|\mathbf{X}_2 - \mathbf{X}_1\|$. Similarly, $\|\mathcal{G}_1(\mathbf{X}_1) - \mathcal{G}_1(\mathbf{X}_2)\| \leq 51\|\mathbf{A}\|_+^2\|\mathbf{B}\|_+^5\|\mathbf{P}^\star\|_+^3\|\mathbf{R}^{-1}\|_+^3\|\mathbf{X}_2 - \mathbf{X}_1\|\epsilon$ and $\|\mathcal{G}_2(\mathbf{X}_1) - \mathcal{G}_2(\mathbf{X}_2)\| \leq 2\|\mathbf{A}\|_+^2\|\mathbf{B}\|_+^6\|\mathbf{P}^\star\|_+^3\|\mathbf{R}^{-1}\|_+^3\|\mathbf{X}_2 - \mathbf{X}_1\|\eta$ can be established. Plugging these results into the relation in (45) shows the bound for $\|\mathcal{K}(\mathbf{X}_1) - \mathcal{K}(\mathbf{X}_2)\|$ in (32) □